Comput. Methods Appl. Mech. Engrg. 198 (2009) 3113–3129

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Discontinuous Galerkin error estimation for linear symmetric hyperbolic systems

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ARTICLE INFO

Article history:
Received 11 November 2008
Accepted 19 May 2009
Available online 3 June 2009

Keywords:
Discontinuous Galerkin method
Linear hyperbolic systems
A posteriori error estimation
Superconvergence

ABSTRACT

In this manuscript we present an error analysis for the discontinuous Galerkin discretization error of multi-dimensional first-order linear symmetric hyperbolic systems of partial differential equations. We perform a local error analysis by writing the local error as a series and showing that its leading term can be expressed as a linear combination of Legendre polynomials of degree p and $p+1$. We apply these asymptotic results and solve relatively small local problems to compute efficient and asymptotically exact estimates of the finite element error. We present computational results for several linear hyperbolic systems in one, two and three dimensions.

1. Introduction

The discontinuous Galerkin (DG) finite element method was first used to solve the neutron equation [27] and then studied for initial-value problems for ordinary differential equations [4,9,27,29,30]. Cockburn and Shu [15] extended the method to solve first-order hyperbolic partial differential equations of conservation laws. They also developed the Local Discontinuous Galerkin (LDG) method for convection–diffusion problems [16]. Consult [14] and the references cited therein for more information on DG methods.

DG methods allow discontinuous bases, which simplify both h-refinement (mesh refinement and coarsening) and p-refinement (method order variation). However, for DG methods to be used in an adaptive framework one needs *a posteriori* error estimates to guide adaptivity and stop the refinement process. The solution space consists of piecewise continuous polynomial functions relative to a structured or unstructured mesh. As such, it can sharply capture solution discontinuities relative to the computational mesh. It maintains local conservation on an elemental basis. The success of the DG method is due to the following properties: (i) does not require continuity across element boundaries, (ii) is locally conservative, (iii) is well suited to solve problems on locally refined meshes with hanging nodes, (iv) exhibits strong superconvergence that can be used to estimate the discretization error, (v) has a simple communication pattern between elements with a common face that makes it useful for parallel computation and (vi) it can handle problems with complex geometries to high order.

A posteriori error estimates lie in the heart of every adaptive finite element algorithm for differential equations. They are used to assess the quality of numerical solutions and guide the adaptive enrichment process where elements having high errors are enriched by h-refinement and/or p-refinement while elements with small errors are h- and/or p-coarsened. Furthermore, error estimates are used to stop the adaptive refinement process. The subject has been active for the last 40 years and has attained a certain level of maturity for diffusive problems [8,33]. Developing accurate and robust *a posteriori* error estimates for hyperbolic problems remains a challenge. An *a posteriori* error analysis for convection and convection-dominated convection–diffusion problems was presented by Johnson and his collaborators [21–23]. More recently, Süli and his collaborators [19,31,32] investigated local and transmitted errors as well as goal-oriented estimates for several numerical methods applied to hyperbolic problems.

However, *a posteriori* error estimation is much less developed for DG methods applied to hyperbolic problems. Several *a posteriori* DG error estimates are known for hyperbolic [12,13,18,26] and diffusive [20,28] problems. The first asymptotically correct *a posteriori* error estimates for hyperbolic problems were developed by Adjerid et al. [4] who constructed the first superconvergence-based *a posteriori* DG error estimates for one-dimensional linear and nonlinear hyperbolic problems. Later, Adjerid and Massey [6,7] showed how to construct accurate error estimates for multi-dimensional problems on rectangular meshes where they showed that the leading term of error is spanned by two $(p+1)$-degree Radau polynomials in the x and y directions, respectively. Krivodonova and Flaherty [25] showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$ and computed DG error estimates by solving local problems involving numerical fluxes, thus requiring information from neighboring *inflow* elements. Adjerid and Baccouch [2,3] investigated DG methods on structured and unstructured

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doi:10.1016/j.cma.2009.05.016

triangular meshes with several finite element spaces to discover new superconvergence properties and compute accurate error estimates. LDG methods for diffusion problems were investigated by Adjerid and Klauser [5] who constructed efficient and accurate *a posteriori* error estimates.

Superconvergence properties for DG methods have been studied in [4,17,24,27] for first-order ordinary differential equations, [4,6,7] for hyperbolic problems and [5,10,11] for diffusion and convection–diffusion problems.

In this manuscript we investigate the superconvergence properties and asymptotically exact estimates of the DG discretization error for first-order linear symmetric hyperbolic systems. We explicitly write down the leading term of the local DG error which is spanned by Legendre polynomials of degree p and $p+1$ when pth-degree polynomial spaces are used. For special hyperbolic systems where the coefficient matrices are nonsingular we show that the leading term of the error is spanned by $(p+1)$th-degree Radau polynomials. We develop an efficient error estimation procedure by solving local finite element problems to compute estimates of the leading term of the error. For smooth solutions we obtain error estimates that converge to the true error under mesh refinement.

This manuscript is organized as follows, In Section 2 we present the problem and its discontinuous Galerkin formulation. In Section 3 we present an error analysis and establish an asymptotic expansion of the local discretization error. In Section 4 we present new superconvergence results and construct an *a posteriori* error estimation procedure. In Section 5 we present several computational results for one-, two- and three-dimensional hyperbolic systems that validate our theory. We conclude with a few remarks and a discussion of our results in Section 6.

2. Discontinuous Galerkin formulation

In this section we introduce a Runge–Kutta discontinuous Galerkin (RKDG) method applied to first-order symmetric linear hyperbolic systems in multiple space dimensions. Let $\mathbf{u}(t,\mathbf{x}) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^m$ be the true solution of the linear hyperbolic system

$$\mathbf{u}_t + \sum_{i=1}^{d} \mathbf{A}_i \mathbf{u}_{,x_i} = \mathbf{f}(t,\mathbf{x}), \quad 0 < t \leqslant T, \quad \mathbf{x} \in \Omega = (0,1)^d, \quad d = 1,2,3 \tag{2.1a}$$

subject to the initial and boundary conditions

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \forall\, \mathbf{x} \in \Omega, \tag{2.1b}$$

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{u}_B(t,\mathbf{x}) \quad \forall\, \mathbf{x} \in \partial\Omega, \quad t \in [0,T], \tag{2.1c}$$

where $\mathbf{A}_i$, $i = 1,\ldots,d$, are $m\times m$ symmetric real matrices. The boundary of Ω is denoted by $\partial\Omega$ and $\mathbf{u}_{,t}$ and $\mathbf{u}_{,x_i}$, respectively, denote the partial derivatives of $\mathbf{u}$ with respect to t and x_i. We select $\mathbf{f}$, the initial conditions $\mathbf{u}_0$ and the boundary conditions $\mathbf{u}_B$ such that $\mathbf{u}(t,\mathbf{x})$ lies in the space $\mathfrak{C}^{p+2} = [C^2([0,T], C^{p+2}(\Omega))]^m$ where

$$C^2([0,T], C^{p+2}(\Omega)) = \left\{ \mathbf{v},\ \middle|\ \frac{\partial^k \mathbf{v}(t,\mathbf{x})}{\partial t^k} \in C^{p+2}(\Omega), \quad k = 0,1,2, 0 < t \leqslant T \right\}. \tag{2.2}$$

The results presented in this manuscript hold for one, two and three space dimensions, however, here we present an error analysis for two-dimensional problems, i.e., $d = 2$, and present computational results for one-, two- and three-dimensional hyperbolic systems.

In order to discretize (2.1) we partition the domain $\Omega = (0,1)^2$ into a uniform mesh consisting of N^2 square elements of size $h = 1/N$ defined as

$$\mathscr{T}_h = \{\omega_{ij} = (ih,(i+1)h) \times (jh,(j+1)h) : 0 \leqslant i,j < N-1\}. \tag{2.3}$$

In our analysis we use the polynomial spaces

$$\mathbb{P}_p = \{\mathbf{v} : \mathbf{v}(\mathbf{x}) = \sum_{i,j} \mathbf{c}_{ij} x_1^i x_2^j,\ \mathbf{c}_{ij} \in \mathbb{R}^m,\ 0 \leqslant i+j \leqslant p\}, \tag{2.4}$$

$$\mathscr{P}_p = \{\mathbf{v} : \mathbf{v}(\mathbf{x}) = \sum_{i,j} \mathbf{c}_{ij} x_1^i x_2^j,\ \mathbf{c}_{ij} \in \mathbb{R}^m,\ \leqslant i,j \leqslant p, \quad 0 \leqslant i+j \leqslant p+1\} \tag{2.5}$$

and the finite element space

$$\mathscr{V}^p = \{\mathbf{v}_h : \mathbf{v}_h|_\omega \in \mathscr{P}_p,\ \omega \in \mathscr{T}_h\}. \tag{2.6}$$

Let $\partial\omega_{ij}$ and $\boldsymbol{\nu}$, respectively, denote the boundary of element ω_{ij} and its unit outward normal vector. Since $\mathbf{v}_h \in \mathscr{V}^p$ may be discontinuous across element boundaries, we define

$$\mathbf{v}_h^+(\mathbf{x}) = \lim_{\epsilon\to 0^+} \mathbf{v}_h(\mathbf{x} - \epsilon\boldsymbol{\nu}), \quad \mathbf{v}_h^-(\mathbf{x}) = \lim_{\epsilon\to 0^+} \mathbf{v}_h(\mathbf{x} + \epsilon\boldsymbol{\nu}) \quad \text{for } \mathbf{x} \in \partial\omega_{ij}. \tag{2.7}$$

The weak discontinuous Galerkin formulation of (2.1) is obtained by multiplying (2.1a) by a test function $\mathbf{v}$, integrating over $\omega_{ij} \in \mathscr{T}_h$ and applying Green's identity to write

$$\iint_{\omega_{ij}} (\mathbf{v}^t \mathbf{u}_{,t} - \mathbf{v}^t_{,x_1} \mathbf{A}_1 \mathbf{u} - \mathbf{v}^t_{,x_2} \mathbf{A}_2 \mathbf{u} - \mathbf{v}^t \mathbf{f})\, d\mathbf{x} + \int_{\partial\omega_{ij}} \mathbf{v}^t (\nu_1 \mathbf{A}_1 + \nu_2 \mathbf{A}_2) \mathbf{u}\, ds = 0. \tag{2.8}$$

We obtain a discrete discontinuous Galerkin finite element formulation of (2.8) by setting $\mathbf{u} = \mathbf{u}_h \in \mathscr{V}^p$, testing against $\mathbf{v}_h \in \mathscr{V}^p$ and selecting a numerical flux to approximate the boundary integral term on $\partial\omega_{ij}$.

Since the matrices $\mathbf{A}_1, \mathbf{A}_2$ are symmetric, they are diagonalizable. Thus, for instance, the matrix $\mathbf{A}_1$ can be factored as

$$\mathbf{A}_1 = \mathbf{P}_1 \mathbf{D} \mathbf{P}_1^t, \quad \mathbf{P}_1 = [\mathbf{P}_{1,1}, \mathbf{P}_{1,2}], \quad \mathbf{P}_{1,1} = [\mathbf{v}_1, \ldots \mathbf{v}_r], \quad \mathbf{P}_{1,2} = [\mathbf{v}_{r+1} \ldots, \mathbf{v}_m], \quad \mathbf{D} = \begin{bmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.9a}$$

where $\mathbf{P}_1$ is the matrix of orthogonal eigenvectors of $\mathbf{A}_1$. We further assume that $\mathbf{A}_1$ is of rank r, *i.e.*, it has r nonzero real eigenvalues ordered as $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0 > \lambda_{k+1} \geqslant \cdots \geqslant \lambda_r, \lambda_{r+1} = \cdots \lambda_m = 0$.

Let us split the $r\times r$ diagonal matrix containing the nonzero eigenvalues of $\mathbf{A}_1$ as $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^+ + \boldsymbol{\Lambda}^-$ where

$$\boldsymbol{\Lambda}^+ = diag(\lambda_1,\ldots,\lambda_k,0,\ldots,0) \quad \text{and} \quad \boldsymbol{\Lambda}^- = diag(0,\ldots,0,\lambda_{k+1},\ldots,\lambda_r). \tag{2.9b}$$

Thus, $\mathbf{A}_1$ can be split as

$$\mathbf{A}_1 = \mathbf{A}_1^+ + \mathbf{A}_1^-, \quad \mathbf{A}_1^+ = \mathbf{P}_1 \mathbf{D}^+ \mathbf{P}_1^t, \quad \mathbf{A}_1^- = \mathbf{P}_1 \mathbf{D}^- \mathbf{P}_1^t, \quad \mathbf{D}^\pm = \begin{bmatrix} \boldsymbol{\Lambda}^\pm & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.10}$$

Therefore $\mathbf{A}_i$, $i = 1, 2$, may be split into a sum of two semidefinite matrices as

$$\mathbf{A}_i = \mathbf{A}_i^+ + \mathbf{A}_i^-, \quad \text{where } \mathbf{A}_i^+ \geqslant 0 \quad \text{and } \mathbf{A}_i^- \leqslant 0. \tag{2.11}$$

Assume that $\mathbf{u}_h = \mathbf{u}_h^+$ is the limit from element ω and $\mathbf{u}^-$ is the limit from the neighboring element $\omega^- \in \mathscr{T}_h$ sharing one edge in $\partial\omega$. In our analysis we need the following functions

$$sign(x) = \begin{cases} + & \text{if } x \geqslant 0 \\ - & \text{if } x < 0, \end{cases} \tag{2.12}$$

which is different from the standard *signum* function defined as

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{otherwise.} \end{cases} \tag{2.13}$$

In our analysis we also need the signum $sgn(\mathbf{A})$ of a symmetric matrix $\mathbf{A}$ given in the following definition:

Definition 1. Let $\mathbf{A}$, for instance, be a symmetric square matrix such that $\mathbf{A} = \mathbf{PDP}^t$, where $\mathbf{D} = diag(\lambda_1, \ldots, \lambda_m)$ and $\mathbf{P}$ is the associated matrix of eigenvectors. Thus, we define its sign matrix as

$$sgn(\mathbf{A}) = \mathbf{P}\ sgn(\mathbf{D})\ \mathbf{P}^t, \tag{2.14}$$

where $sgn(\mathbf{D}) = diag(sgn(\lambda_1), \ldots, sgn(\lambda_m))$

Now we are ready to define the upwind flux $\mathscr{A}(\mathbf{u}_h^-, \mathbf{u}_h^+)$ on $\partial\omega_{ij}$ as

$$\mathscr{A}(\mathbf{u}_h^-, \mathbf{u}_h^+) = (\nu_1 \mathbf{A}_1^{\mu_1} + \nu_2 \mathbf{A}_2^{\mu_2})\mathbf{u}_h^+ + (\nu_1 \mathbf{A}_1^{\bar{\mu}_1} + \nu_2 \mathbf{A}_2^{\bar{\mu}_2})\mathbf{u}_h^- \quad \forall\ \mathbf{x} \in \partial\omega_{ij}, \tag{2.15}$$

where μ_i and $\bar{\mu}_i$, respectively, denote $sign(\nu_i)$ and $sign(-\nu)$.

The discontinuous Galerkin method consists of finding $\mathbf{u}_h \in \mathscr{V}^p$ such that on each element $\omega_{ij} \in \mathscr{T}_h$

$$\iint_{\omega_{ij}} [\mathbf{v}_h^t \mathbf{u}_{h,t} - \mathbf{v}_{h,x_1}^t \mathbf{A}_1 \mathbf{u}_h - \mathbf{v}_{h,x_2}^t \mathbf{A}_2 \mathbf{u}_h - \mathbf{v}_h^t \mathbf{f}]\, d\mathbf{x} + \int_{\partial\omega_{ij}} \mathbf{v}_h^t (\nu_1 \mathbf{A}_1^{\mu_1} + \nu_2 \mathbf{A}_2^{\mu_2})\mathbf{u}_h + \mathbf{v}_h^t (\nu_1 \mathbf{A}_1^{\bar{\mu}_1} + \nu_2 \mathbf{A}_2^{\bar{\mu}_2})\mathbf{u}_h^-\, ds = 0 \quad \forall\ \mathbf{v}_h \in \mathscr{P}_p, \tag{2.16}$$

where we used $\mathbf{u}_h^+|_{\partial\omega_{ij}} = \mathbf{u}_h|_{\partial\omega_{ij}}$.

Applying Green's theorem we obtain the equivalent DG formulation which consists of finding $\mathbf{u}_h \in \mathscr{V}^p$ such that

$$\iint_{\omega_{ij}} \mathbf{v}_h^t (\mathbf{u}_{h,t} + \mathbf{A}_1 \mathbf{u}_{h,x_1} + \mathbf{A}_2 \mathbf{u}_{h,x_2} - \mathbf{f})\, d\mathbf{x} + \int_{\partial\omega_{ij}} \mathbf{v}_h^t (\nu_1 \mathbf{A}_1^{\bar{\mu}_1} + \nu_2 \mathbf{A}_2^{\bar{\mu}_2})(\mathbf{u}_h^- - \mathbf{u}_h)\, ds = 0 \quad \forall\ \mathbf{v}_h \in \mathscr{P}_p. \tag{2.17}$$

We will approximate the initial and boundary conditions $\mathbf{u}_0$ and $\mathbf{u}_B$ by functions in $\mathscr{V}^p$ to compute initial values for the resulting time-dependent ODE system.

For the purpose of analyzing the behavior of the spatial discretization error, we assume the integration in time to be exact. In our numerical experiments of Section 5, we use a temporal error tolerance much smaller than the spatial error tolerance by applying a high-order Runge–Kutta method to solve the resulting ODE system.

3. Local error analysis

In our error analysis we need Legendre polynomials defined, for instance, by Rodrigues formula [1]

$$\widetilde{L}_p(x) = \frac{1}{2^p p!} \frac{d^p}{dx^p}[(x^2 - 1)^p], \qquad -1 \leqslant x \leqslant 1, \qquad p = 0, 1, \ldots. \tag{3.1a}$$

The shifted pth-degree Legendre polynomials on $[0,1]$ denoted by $L_p(\xi), 0 \leqslant \xi \leqslant 1,\ p = 0, 1, \ldots$ are orthogonal to all polynomials of degree not exceeding $p-1$ and satisfy the following properties

$$\int_0^1 L_p(\xi) L'_{p+1}(\xi) d\xi = 2, \qquad \int_0^1 L_k(\xi) L_j(\xi) d\xi = \frac{\delta_{jk}}{2j+1}, \tag{3.1b}$$

where δ_{jk} is the Kronecker delta equal to 1 if $j = k$ and 0 otherwise.

We also need the pth-degree right Radau polynomial and left Radau polynomial, respectively, defined by $R_p^+(\xi) = L_p(\xi) - L_{p-1}(\xi)$ and $R_p^-(\xi) = L_p(\xi) + L_{p-1}(\xi)$.

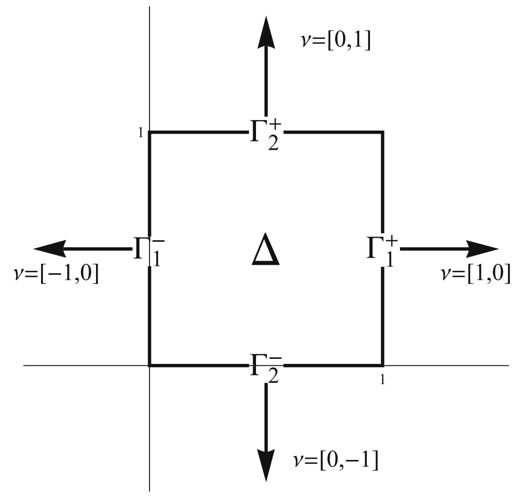

Fig. 1. The reference element $\Delta = [0,1]^2$ with boundary and outer normal unit vectors.

In the remainder of this manuscript we present a local error analysis on the element $\omega = (0,h)^2$ mapped to the reference element $\Delta = (0,1)^2$ shown in Fig. 1. Let $\Gamma = \partial\Delta$ denote the boundary of Δ which can be split into $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \Gamma_1^- \cup \Gamma_1^+$, $\Gamma_2 = \Gamma_2^- \cup \Gamma_2^+$ such that Γ_1^- and Γ_1^+, respectively, denote the left and right edges of Δ, while Γ_2^- and Γ_2^+, respectively, denote the bottom and top edges of Δ shown in Fig. 1.

Let the affine transformation from Δ onto ω be $\mathbf{x}(\xi) = (h\xi_1, h\xi_2)$, and

$$\partial\omega = \gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^-, \tag{3.2}$$

where $\gamma_i^s = \mathbf{x}(\Gamma_i^s),\ s = +,-,\ i = 1,2$. For instance, $\gamma_1^+ = \{(h,y),\ 0 < y < h\}$.

In order to approximate the initial and boundary conditions by functions in $\mathscr{P}_p$ on every element and its boundary edges such that the resulting approximation error is consistent with the discontinuous Galerkin discretization error presented in Theorem 3.1, we define few special approximation operators.

First, let $\mathbf{w} \in [C^{p+2}([0,h]^2)]^m$ and consider its $(p+1)$th-degree Taylor polynomial about $\mathbf{x} = 0$

$$T_{p+1}\mathbf{w}(\mathbf{x}) = \sum_{|\alpha| \leqslant p+1} \frac{D^\alpha \mathbf{w}(0)}{\alpha!} \mathbf{x}^\alpha, \tag{3.3a}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2, \alpha! = \alpha_1!\alpha_2!, \mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$ and the differential operator $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$.

If a_{p+1} denotes the coefficient of ξ^{p+1} in $L_{p+1}(\xi)$, then we consider the approximation operator $\pi : [C^{p+2}([0,h]^2)]^m \to \mathscr{P}_p$ defined as

$$\pi\mathbf{w}(\mathbf{x}) = T_{p+1}\mathbf{w}(\mathbf{x}) - h^{p+1} \sum_{j=1}^{2} \left(L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j - L_p\left(\frac{x_j}{h}\right)\mathbf{d}_j \right), \tag{3.3b}$$

where

$$\mathbf{c}_j = \frac{1}{a_{p+1}} \frac{1}{(p+1)!} \frac{\partial^{p+1}\mathbf{w}(0)}{\partial x_j^{p+1}} \quad \text{and} \quad \mathbf{d}_j = sgn(\mathbf{A}_j)\mathbf{c}_j, \quad j = 1, 2. \tag{3.3c}$$

We approximate the boundary conditions on Γ_1 using

$$\pi_1\mathbf{w}(a, x_2) = \sum_{k=0}^{p+1} \frac{D^{(0,k)}\mathbf{w}(0,0)}{k!} x_2^k - h^{p+1}\left(L_{p+1}\left(\frac{x_2}{h}\right)\mathbf{c}_2 - L_p\left(\frac{x_2}{h}\right)\mathbf{d}_2 \right), \quad 0 \leqslant x_2 \leqslant h, \tag{3.4a}$$

where $a = 0$ for γ_1^- and $a = h$ for γ_1^+.

On γ_2 we apply

$$\pi_2\mathbf{w}(x_1,b) = \sum_{k=0}^{p+1}\frac{D^{(k,0)}\mathbf{w}(0,0)}{k!}x_1^k - h^{p+1}\left(L_{p+1}\left(\frac{x_1}{h}\right)\mathbf{c}_1 - L_p\left(\frac{x_1}{h}\right)\mathbf{d}_1\right),\quad 0 \leqslant x_1 \leqslant h, \tag{3.4b}$$

where $b = 0$ for γ_2^- and $b = h$ for γ_2^+.

We note that the coefficients $\mathbf{c}_j$ and $\mathbf{d}_j$ in the boundary operators π_1 and π_2 satisfy the conditions (3.3c).

Now we are ready to state and prove several preliminary lemmas.

Lemma 3.1. *Let $\omega = (0,h)^2, \mathbf{w}(\mathbf{x}) \in C^{p+2}(\bar{\omega})$ and $\mathbf{x} : \Delta \to \omega$ be the linear mapping $\mathbf{x} = h\xi$. Let $\pi\mathbf{w} \in \mathscr{P}_p$ on ω and $\pi_i\mathbf{w} \in \mathscr{P}_p$ on the boundary $\gamma_i^+ \cup \gamma_i^-$, $i = 1,2$, defined by (3.3) and (3.4), respectively. Then there exists a positive constant C independent of h such that the l^2 norm of the interpolation errors can be bounded as*

$$\sup_{\xi\in\Delta}\|\mathbf{w}(\mathbf{x}(\xi)) - \pi\mathbf{w}(\mathbf{x}(\xi)) - h^{p+1}\sum_{j=1}^{2}(L_{p+1}(\xi_j)\mathbf{c}_j - L_p(\xi_j)\mathbf{d}_j)\|_2 \leqslant Ch^{p+2}, \tag{3.5}$$

$$\sup_{\xi\in\Gamma_1}\|\mathbf{w}(\mathbf{x}(\xi)) - \pi_1\mathbf{w}(\mathbf{x}(\xi)) - h^{p+1}(L_{p+1}(\xi_2)\mathbf{c}_2 - L_p(\xi_2)\mathbf{d}_2)\|_2 \leqslant Ch^{p+2}, \tag{3.6}$$

$$\sup_{\xi\in\Gamma_2}\|\mathbf{w}(\mathbf{x}(\xi)) - \pi_2\mathbf{w}(\mathbf{x}(\xi)) - h^{p+1}(L_{p+1}(\xi_1)\mathbf{c}_1 - L_p(\xi_1)\mathbf{d}_1)\|_2 \leqslant Ch^{p+2}, \tag{3.7}$$

where $\mathbf{c}_i$ and $\mathbf{d}_i$ satisfy

$$\mathbf{A}_i^+\mathbf{c}_i = \mathbf{A}_i^+\mathbf{d}_i \quad \text{and } \mathbf{A}_i^-\mathbf{c}_i = -\mathbf{A}_i^-\mathbf{d}_i. \tag{3.8}$$

Proof. Applying Taylor's theorem to $\mathbf{w} \in [C^{p+2}([0,h]^2)]^m$ yields

$$\mathbf{w}(\mathbf{x}) = T_{p+1}\mathbf{w}(\mathbf{x}) + \sum_{|\alpha|=p+2}R_\alpha(\mathbf{x})\mathbf{x}^\alpha, \tag{3.9}$$

where the remainder can be bounded on $\bar{\omega} = [0,h]^2$ as

$$\|\mathbf{w}(\mathbf{x}) - T_{p+1}\mathbf{w}(\mathbf{x})\|_2 = \left\|\sum_{|\alpha|=p+2}R_\alpha(\mathbf{x})\mathbf{x}^\alpha\right\|_2 \leqslant \sum_{|\alpha|=p+2}\|R_\alpha(\mathbf{x})\|_2 h^{p+2} \leqslant Ch^{p+2}. \tag{3.10}$$

From the definition of $\pi\mathbf{w}$ in Eq. (3.3) we observe that

$$\mathbf{w}(\mathbf{x}(\xi)) - \pi\mathbf{w}(\mathbf{x}(\xi)) - h^{p+1}\sum_{j=1}^{2}(L_{p+1}(\xi_j)\mathbf{c}_j - L_p(\xi_j)\mathbf{d}_j) = \mathbf{w}(\mathbf{x}) - T_{p+1}\mathbf{w}(\mathbf{x}). \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) we establish (3.5).

Finally, following the same line of reasoning we establish (3.6) and (3.7) which concludes the proof. □

The approximation $\pi\mathbf{w}$ defined in (3.3) is only used for the analysis. In practice we use a corrected L^2 projection onto $\mathbb{P}_{p+1}$ consistent with the discontinuous Galerkin approximation.

If $\tilde{\pi}\mathbf{w}$ is the L^2 projection of $\mathbf{w}$ onto $\mathbb{P}_{p+1}$ we define on Δ

$$\pi\mathbf{w} = \tilde{\pi}\mathbf{w} - (L_{p+1}(\xi_1) - sgn(\mathbf{A}_1)L_p(\xi_1))\mathbf{c}_{p+1,0} - (L_{p+1}(\xi_2) - sgn(\mathbf{A}_2)L_p(\xi_2))\mathbf{c}_{0,p+1}, \tag{3.12a}$$

where

$$\tilde{\pi}\mathbf{w} = \sum_{0\leqslant i+j\leqslant p+1}\mathbf{c}_{i,j}L_i(\xi_i)L_j(\xi_j), \quad \mathbf{c}_{i,j} = \frac{\int_\Delta \mathbf{w}(\xi)L_i(\xi_i)L_j(\xi_j)d\xi}{\int_\Delta L_i(\xi_i)^2L_j(\xi_j)^2d\xi}. \tag{3.12b}$$

In our numerical experiments we also use the L^2 projection $\Pi\mathbf{w}$ of $\mathbf{w}$ onto $\mathscr{P}_p$. Using the set of orthogonal functions $\{\psi_{ij}(\xi_1,\xi_2) = L_i(\xi_1)L_j(\xi_2),\ 0 \leqslant i,j \leqslant p,\ 0 \leqslant i+j \leqslant p+1\}$, $\Pi\mathbf{w}$ may be defined as

$$\Pi\mathbf{w}(\mathbf{x}(\xi)) = \sum_{\substack{0\leqslant i,j\leqslant p\\ 0\leqslant i+j\leqslant p+1}}\frac{\int\int_\Delta \mathbf{w}(\mathbf{x}(\xi))\psi_{i,j}(\xi_1,\xi_2)\,d\xi}{\int\int_\Delta \psi_{ij}^2(\xi_1,\xi_2)\,d\xi}\psi_{ij}(\xi_1,\xi_2). \tag{3.13}$$

Noting that $\mathbb{P}_p \subseteq \mathscr{P}_p$, we have the standard *a priori* bound

$$\sup_{\mathbf{x}\in\omega}\|\mathbf{w}(\mathbf{x}) - \Pi\mathbf{w}(\mathbf{x})\|_2 \leqslant Ch^{p+1}. \tag{3.14}$$

For $j = 1,2$, let j' denote the dual of j defined as

$$j' = \begin{cases} 2 & \text{if } j = 1,\\ 1 & \text{if } j = 2.\end{cases} \tag{3.15}$$

Let $\mathscr{N}(A)$ and $\mathscr{R}(A)$, respectively, denote and the null space and range of $\mathbf{A}$.

In the next lemma we prove a linear algebra result needed in our error analysis.

Lemma 3.2. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be symmetric matrices such that the $m \times (m-r)$ matrix $\mathbf{P}_{j,2}$, $j = 1,2$, denotes the matrix of all $(m-r)$ orthogonal eigenvectors associated with the zero eigenvalue of $\mathbf{A}_j$. If the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy either of the following assumptions*

1. $\mathbf{A}_j$ is invertible for $j = 1$ or $j = 2$,
2. $\mathscr{N}(\mathbf{P}_{j,2}^t\mathbf{A}_{j'}\mathbf{P}_{j,2}) = \{0\}$ for $j = 1$ or $j = 2$,

then $\mathscr{N}(A_1)\bigcap\mathscr{N}(A_2) = \{0\}$.

Proof. If either $\mathbf{A}_1$ or $\mathbf{A}_2$ is invertible, the proof is straight forward.

Now, let us prove the lemma if one of the remaining conditions are satisfied. Without loss of generality, we assume $j = 1$ and let $\mathbf{q}$ be an arbitrary vector in $\mathscr{N}(\mathbf{A}_1)\bigcap\mathscr{N}(\mathbf{A}_2)$. Thus, $\mathbf{A}_1\mathbf{q} = 0$ and $\mathbf{A}_2\mathbf{q} = 0$ which are equivalent to

$$\mathbf{P}_1^t\mathbf{A}_1\mathbf{P}_1\mathbf{w} = \begin{pmatrix}\mathbf{\Lambda} & 0\\ 0 & 0\end{pmatrix}\mathbf{w} = 0 \quad \text{and } \mathbf{P}_1^t\mathbf{A}_2\mathbf{P}_1\mathbf{w} = 0, \tag{3.16}$$

where $\mathbf{w} = \mathbf{P}_1^t\mathbf{q}$ with $\mathbf{P}_1 = [P_{1,1}, P_{1,2}]$ containing the eigenvectors of $\mathbf{A}_1$.

Thus, if we split $\mathbf{w}$ as $\mathbf{w} = \begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix}$, where $\mathbf{w}_1 \in \mathbb{R}^r$, then $\mathbf{\Lambda}\mathbf{w}_1 = 0$ yields $\mathbf{w}_1 = 0$.

On the other-hand $\mathbf{P}_1^t\mathbf{A}_2\mathbf{P}_1\mathbf{w} = 0$ yields $\mathbf{P}_{1,2}^t\mathbf{A}_2\mathbf{P}_{1,2}\mathbf{w}_2 = 0$ which, in turns, leads to $\mathbf{w}_2 = 0$. Therefore, $\mathbf{w} = \mathbf{q} = 0$.

Here we note that the converse of the previous lemma is not true.

Lemma 3.3. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be symmetric matrices satisfying one of the assumptions of* Lemma 3.2 *and let $\boldsymbol{\nu} = (\nu_1,\nu_2)^t$ denote the unit normal vector to the square $\Delta = [0,1]^2$.If $\mathbf{q} \in \mathscr{P}_p$ satisfies*

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{q}_{,\xi_1} + \mathbf{A}_2\mathbf{q}_{,\xi_2})\,d\xi - \int_\Gamma \mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1} + \nu_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}\,ds = 0 \quad \forall\, \mathbf{v} \in \mathscr{P}_p \tag{3.17}$$

then, $\mathbf{q} = 0$ on Δ.

Proof. The proof follows the same line of reasoning for all possible cases. Here, we present the proof for assumption 2 of Lemma 3.2 with $j = 1$ only.

First we integrate (3.17) by parts to obtain

$$-\iint_\Delta (\mathbf{v}_{,\xi_1}^t\mathbf{A}_1 + \mathbf{v}_{,\xi_2}^t\mathbf{A}_2)\mathbf{q}\,d\xi + \int_\Gamma \mathbf{v}^t(\nu_1\mathbf{A}_1^{\mu_1} + \nu_2\mathbf{A}_2^{\mu_2})\mathbf{q}\,ds = 0. \tag{3.18}$$

Adding (3.17) and (3.18) and testing against $\mathbf{v} = \mathbf{q}$ we note that by the symmetry of $\mathbf{A}_1$ and $\mathbf{A}_2$, the double integrals on Δ cancel out. Thus, $\mathbf{q}$ satisfies

$$\int_{\Gamma_1} \mathbf{q}^t \nu_1(\mathbf{A}_1^{\mu_1} - \mathbf{A}_1^{\bar{\mu}_1})\mathbf{q}\,ds + \int_{\Gamma_2} \mathbf{q}^t \nu_2(\mathbf{A}_2^{\mu_2} - \mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}\,ds = \int_{\Gamma_1} \mathbf{q}^t(\mathbf{A}_1^+ - \mathbf{A}_1^-)\mathbf{q}\,ds + \int_{\Gamma_2} \mathbf{q}^t(\mathbf{A}_2^+ - \mathbf{A}_2^-)\mathbf{q}\,ds = 0. \quad (3.19)$$

Since $(\mathbf{A}_i^+ - \mathbf{A}_i^-)$ is symmetric positive semi-definite it admits a Cholesky factorization $(\mathbf{A}_i^+ - \mathbf{A}_i^-) = \mathbf{L}_i^t\mathbf{L}_i$. Thus, (3.19) can be written as

$$\int_{\Gamma_1} \|\mathbf{L}_1\mathbf{q}\|_2^2\,ds + \int_{\Gamma_2} \|\mathbf{L}_2\mathbf{q}\|_2^2\,ds = 0. \quad (3.20)$$

Thus, $\mathbf{L}_i\mathbf{q} = 0$ on Γ_i which yields

$$\mathbf{L}_i^t(\mathbf{L}_i\mathbf{q}) = (\mathbf{A}_i^+ - \mathbf{A}_i^-)\mathbf{q} = 0 \quad \text{on } \Gamma_i, \quad i = 1,2. \quad (3.21)$$

Since $\mathcal{N}(\mathbf{A}_i^+ - \mathbf{A}_i^-) = \mathcal{N}(\mathbf{A}_i^+ + \mathbf{A}_i^-)$, (3.21) leads to $(\mathbf{A}_i^+ + \mathbf{A}_i^-)\mathbf{q}|_{\Gamma_i} = 0$, $i = 1, 2$. Thus, one can easily show that

$$\mathbf{A}_i^{\pm}\mathbf{q}|_{\Gamma_i} = 0, \quad i = 1,2. \quad (3.22)$$

Testing against $\mathbf{v} = \mathbf{A}_1\mathbf{q}_{,\xi_1} + \mathbf{A}_2\mathbf{q}_{,\xi_2}$ in (3.17) and combining the resulting equation with (3.22) lead to

$$\iint_\Delta \|\mathbf{A}_1\mathbf{q}_{\xi_1} + \mathbf{A}_2\mathbf{q}_{\xi_2}\|_2^2 \quad d\xi = 0, \quad (3.23)$$

which in turn yields

$$\mathbf{A}_1\mathbf{q}_{\xi_1} + \mathbf{A}_2\mathbf{q}_{\xi_2} = 0 \quad \forall\ \xi\ \in\ \Delta. \quad (3.24)$$

Next, we let

$$\mathbf{B}_1 = \mathbf{P}_1^t\mathbf{A}_1\mathbf{P}_1 = \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and } \mathbf{B}_2 = \mathbf{P}_1^t\mathbf{A}_2\mathbf{P}_1 = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \quad (3.25)$$

where $\mathbf{P}_1 = [\mathbf{P}_{1,1}, \mathbf{P}_{1,2}]$ and $\mathbf{B}_{ij} = \mathbf{P}_{1,i}^t\mathbf{A}_2\mathbf{P}_{1,j}$, $i,j = 1,2$.

Applying $\mathbf{A}_1\mathbf{q}|_{\Gamma_1} = 0$ we write

$$\mathbf{B}_1\mathbf{w}|_{\Gamma_1} = \Lambda\mathbf{w}_1|_{\Gamma_1} = 0, \quad (3.26)$$

where $\mathbf{w} = \mathbf{P}^t\mathbf{q} = (\mathbf{w}_1, \mathbf{w}_2)^t$. This establishes $\mathbf{w}_1|_{\Gamma_1} = 0$.

On the other-hand $\mathbf{A}_2\mathbf{q}|_{\Gamma_2} = 0$ leads to

$$\mathbf{B}_2\mathbf{w}|_{\Gamma_2} = 0, \quad (3.27)$$

which can be split as

$$(\mathbf{B}_{11}\mathbf{w}_1 + \mathbf{B}_{12}\mathbf{w}_2)|_{\Gamma_2} = 0, \quad (3.28a)$$

$$(\mathbf{B}_{21}\mathbf{w}_1 + \mathbf{B}_{22}\mathbf{w}_2)|_{\Gamma_2} = 0. \quad (3.28b)$$

Pre-multiplying (3.24) by $\mathbf{P}^t$ yields

$$\Lambda\mathbf{w}_{1,\xi_1} + \mathbf{B}_{11}\mathbf{w}_{1,\xi_2} + \mathbf{B}_{12}\mathbf{w}_{2,\xi_2} = 0, \quad (3.29a)$$

$$\mathbf{B}_{21}\mathbf{w}_{1,\xi_2} + \mathbf{B}_{22}\mathbf{w}_{2,\xi_2} = 0 \quad \forall\ \xi \in \Delta. \quad (3.29b)$$

Since $\mathbf{B}_{22}$ is invertible, we can express $\mathbf{w}_{2,\xi_2}$ as

$$\mathbf{w}_{2,\xi_2} = -\mathbf{B}_{22}^{-1}\mathbf{B}_{21}\mathbf{w}_{1,\xi_2}, \quad (3.30a)$$

which leads to

$$\Lambda\mathbf{w}_{1,\xi_1} + (\mathbf{B}_{11} - \mathbf{B}_{12}\mathbf{B}_{22}^{-1}\mathbf{B}_{21})\mathbf{w}_{1,\xi_2} = 0 \quad \forall\ \xi \in \Delta. \quad (3.30b)$$

Combining the fact that $\mathbf{w}_1|_{\Gamma_1} = 0$ and (3.30b) we establish that

$$\frac{\partial^k \mathbf{w}_1}{\partial\xi_1^l\partial\xi_2^{k-l}} = 0 \quad \forall\ \xi \in \Delta,\ k \geqslant 0, \quad l = 0, \cdots, k. \quad (3.31)$$

Since $\mathbf{w}_1$ is a polynomial, $\mathbf{w}_1 = 0$.

Combining (3.28b) and (3.30a) we establish that $\mathbf{w}_2|_\Delta = 0$. Thus, $\mathbf{q} = 0\ \forall\ \xi \in \Delta$. which completes the proof of the lemma. □

Now we are ready to state the main theorem for the local spatial discretization error.

Theorem 3.1. *Let $\mathbf{A}_i$, $i = 1, 2$, be symmetric matrices satisfying the conditions of Lemma 3.3 and let $\mathbf{u} \in \mathfrak{C}^{p+2}$ and $\mathbf{u}_h \in \mathscr{P}_p$ be the solutions of (2.1) and (2.17), respectively, on $\omega = (0, h)^2$. Here $\mathbf{u}_h$ is computed by approximating the boundary conditions as*

$$\mathbf{u}_h^-(t, \mathbf{x}) = \pi_i\mathbf{u}_B(t, \mathbf{x}), \quad t > 0, \quad \mathbf{x} \in \gamma_i^+ \cup \gamma_i^-, \quad i = 1,2 \quad (3.32)$$

and use initial conditions as either $\pi\mathbf{u}_0$ or $\Pi\mathbf{u}_0$. Then, the local finite element error on ω for $t = \mathcal{O}(1)$ and $p > 0$ can be written as

$$\mathbf{e}(t, h\xi) = \mathbf{u}(t, h\xi) - \mathbf{u}_h(t, h\xi) = h^{p+1}\sum_{j=1}^{2}(L_{p+1}(\xi_j)\mathbf{c}_j(t) - L_p(\xi_j)\mathbf{d}_j(t)) + \mathcal{O}(h^{p+2}), \quad (3.33a)$$

where

$$\mathbf{A}_i^+\mathbf{c}_i(t) - \mathbf{A}_i^+\mathbf{d}_i(t) = 0 \quad \mathbf{A}_i^-\mathbf{c}_i(t) + \mathbf{A}_i^-\mathbf{d}_i(t) = 0, \quad i = 1,2. \quad (3.33b)$$

Proof. Subtracting the weak DG formulation (2.17) from (2.1a), the local discretization error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ satisfies the DG orthogonality condition

$$\iint_\omega \mathbf{v}^t(\mathbf{e}_{,t} + \mathbf{A}_1\mathbf{e}_{,x_1} + \mathbf{A}_2\mathbf{e}_{,x_2})\,d\mathbf{x} + \int_{\partial\omega} \mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1} + \nu_2\mathbf{A}_2^{\bar{\mu}_2})(\mathbf{e}^- - \mathbf{e})\,ds = 0 \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \quad (3.34)$$

Apply the scalings $\tau = \frac{t}{T}$ and $\xi = \frac{\mathbf{x}}{h}$ to write $\hat{\mathbf{e}}(\tau, \xi) = \mathbf{e}(T\tau, h\xi)$ to obtain

$$\iint_\Delta \mathbf{v}^t\left(\frac{h}{T}\hat{\mathbf{e}}_{,\tau} + \mathbf{A}_1\hat{\mathbf{e}}_{,\xi_1} + \mathbf{A}_2\hat{\mathbf{e}}_{,\xi_2}\right)d\xi + \int_\Gamma \mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1} + \nu_2\mathbf{A}_2^{\bar{\mu}_2})(\hat{\mathbf{e}}^- - \hat{\mathbf{e}})\,ds = 0 \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \quad (3.35)$$

In order to investigate the asymptotic behavior of the local error, we start by writing the Maclaurin series of $\hat{\mathbf{e}}$ with respect to the mesh parameter h as

$$\hat{\mathbf{e}}(\tau, \xi) = \sum_{k=0}^{p+1} h^k\mathbf{q}_k(\tau, \xi) + \mathcal{O}(h^{p+2}), \quad (3.36)$$

where

$$\mathbf{q}_k(\tau, \xi) = \frac{1}{k!}\left.\frac{d^k(\mathbf{u} - \mathbf{u}_h)(\tau T, \xi h, h)}{dh^k}\right|_{h=0}. \quad (3.37)$$

We used the fact that $\mathbf{u}_h(t, \mathbf{x}, h)$ is a function of t, $\mathbf{x}$ and h.

Next, from the definition of $\pi_i\mathbf{u}_B$ in Lemma 3.1, $\hat{\mathbf{e}}^-$ satisfies

$$\hat{\mathbf{e}}^-(\tau, \xi) = (\mathbf{u}_B - \pi_1\mathbf{u}_B)(\tau, \xi) = \mathbf{r}_2(\tau, \xi_2)h^{p+1} + \mathcal{O}(h^{p+2}) \quad \text{for } \xi \in \Gamma_1, \quad (3.38a)$$

$$\hat{\mathbf{e}}^-(\tau, \xi) = (\mathbf{u}_B - \pi_2\mathbf{u}_B)(\tau, \xi) = \mathbf{r}_1(\tau, \xi_1)h^{p+1} + \mathcal{O}(h^{p+2}) \quad \text{for } \xi \in \Gamma_2, \quad (3.38b)$$

where

$$\mathbf{r}_i(\tau, \xi_i) = L_{p+1}(\xi_i)\hat{\mathbf{c}}_i(\tau) - L_p(\xi_i)\hat{\mathbf{d}}_i(\tau) \quad (3.38c)$$

with

$$\hat{\mathbf{c}}_i(\tau) = \mathbf{c}_i(t)|_{t=\tau T} = \frac{1}{a_{p+1}}\frac{1}{(p+1)!}\left.\frac{\partial^{p+1}}{\partial x_i^{p+1}}\mathbf{u}(t, 0)\right|_{t=\tau T}, \quad (3.38d)$$

$$\mathbf{A}_i^+\hat{\mathbf{c}}_i(\tau) - \mathbf{A}_i^+\hat{\mathbf{d}}_i(\tau) = 0 \quad \text{and } \mathbf{A}_i^-\hat{\mathbf{c}}_i(\tau) + \mathbf{A}_i^-\hat{\mathbf{d}}_i(\tau) = 0. \quad (3.38e)$$

Substituting (3.36), (3.38a) and (3.38b) in (3.35) yields

$$\sum_{k=0}^{p+1} h^k \iint_\Delta \mathbf{v}^t\left(\frac{h}{T}\mathbf{q}_{k,\tau} + \mathbf{A}_1\mathbf{q}_{k,\xi_1} + \mathbf{A}_2\mathbf{q}_{k,\xi_2}\right)d\xi - \int_\Gamma \mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1} + \nu_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_k\,ds = -h^{p+1}\left(\int_{\Gamma_1} \mathbf{v}^t\nu_1\mathbf{A}_1^{\bar{\mu}_1}\mathbf{r}_2\,ds + \int_{\Gamma_2} \mathbf{v}^t\nu_2\mathbf{A}_2^{\bar{\mu}_2}\mathbf{r}_1\,ds\right) + \mathcal{O}(h^{p+2}). \quad (3.39)$$

Now, we will use induction to prove that $\mathbf{q}_k = 0,\ k = 0, 1, \ldots, p$ by first assuming that $T = \mathcal{O}(1)$ and setting to zero all terms having the same power of h. Thus, the $\mathcal{O}(1)$ term $\mathbf{q}_0$ satisfies the orthogonality condition

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{q}_{0,\xi_1} + \mathbf{A}_2\mathbf{q}_{0,\xi_2})\,d\xi - \int_\Gamma \mathbf{v}^t(v_1\mathbf{A}_1^{\bar{\mu}_1} + v_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_0\,ds = 0 \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.40}$$

By Lemma 3.3, $\mathbf{q}_0 = 0$.

Now assume that $\mathbf{q}_j = 0,\ j = 0, 1, \ldots, k-1 < p$. Thus, the $\mathcal{O}(h^k)$ term is written as

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{q}_{k,\xi_1} + \mathbf{A}_2\mathbf{q}_{k,\xi_2})\,d\xi - \int_\Gamma \mathbf{v}^t(v_1\mathbf{A}_1^{\bar{\mu}_1} + v_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_k\,ds = 0 \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.41}$$

By Lemma 3.3, $\mathbf{q}_k = 0$ on Δ. Thus, by induction, $\mathbf{q}_k = 0,\ k = 0, \ldots, p$.

The $\mathcal{O}(h^{p+1})$ term satisfies the orthogonality condition

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{q}_{p+1,\xi_1} + \mathbf{A}_2\mathbf{q}_{p+1,\xi_2})\,d\xi = \int_{\Gamma_1} \mathbf{v}^t v_1\mathbf{A}_1^{\bar{\mu}_1}(\mathbf{q}_{p+1} - \mathbf{r}_2)\,ds + \int_{\Gamma_2} \mathbf{v}^t v_2\mathbf{A}_2^{\bar{\mu}_2}(\mathbf{q}_{p+1} - \mathbf{r}_1)\,ds. \tag{3.42}$$

By Eq. (3.36),

$$\mathbf{q}_{p+1}(\tau,\xi) = \frac{1}{(p+1)!}\left.\frac{d^{p+1}(\mathbf{u}-\mathbf{u}_h)(\tau T, \xi h)}{dh^{p+1}}\right|_{h=0} \tag{3.43}$$

$$= \sum_{|\alpha|=p+1} \frac{1}{\alpha!} D^\alpha(\mathbf{u}-\mathbf{u}_h)(\tau T, 0)\xi^\alpha \tag{3.44}$$

$$= \sum_{i=1}^{2} \frac{1}{(p+1)!}\frac{\partial^{p+1}}{\partial x_i^{p+1}}(\mathbf{u}-\mathbf{u}_h)(\tau T, 0)\xi_i^{p+1} + \mathbf{p}_1(\tau,\xi), \tag{3.45}$$

where, for a fixed $\tau, \mathbf{p}_1(\tau,\xi) \in \mathscr{P}_p$. By Eq. (3.38c)

$$\sum_{i=1}^{2} \mathbf{r}_i(\tau,\xi_i) = \sum_{i=1}^{2} (L_{p+1}(\xi_i)\hat{\mathbf{c}}_i - L_p(\xi_i)\hat{\mathbf{d}}_i) \tag{3.46}$$

$$= \sum_{i=1}^{2} \frac{1}{(p+1)!}\frac{\partial^{p+1}}{\partial x_i^{p+1}}\mathbf{u}(\tau T, 0)\xi_i^{p+1} + \mathbf{p}_2(\tau,\xi), \tag{3.47}$$

where $\mathbf{p}_2 \in \mathscr{P}_p$. We further note that since $\mathbf{u}_h \in \mathscr{P}_p, \frac{\partial^{p+1}\mathbf{u}_h}{\partial x_i^{p+1}} = 0$, which in turn leads to

$$\mathbf{q}_{p+1}(\tau,\xi) - \mathbf{r}_1(\tau,\xi_1) - \mathbf{r}_2(\tau,\xi_2) = \mathbf{p}_1(\tau,\xi) - \mathbf{p}_2(\tau,\xi) = \mathbf{p}(\tau,\xi) \ \in \mathscr{P}_p. \tag{3.48}$$

Noting that $\mathbf{r}_2$ is independent of ξ_1 and $\mathbf{r}_1$ is independent of ξ_2, solving (3.48) for $\mathbf{q}_{p+1}$ and substituting into (3.42) yields

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1(\mathbf{p}+\mathbf{r}_1)_{,\xi_1} + \mathbf{A}_2(\mathbf{p}+\mathbf{r}_2)_{,\xi_2})\,d\xi = \int_{\Gamma_1} \mathbf{v}^t v_1\mathbf{A}_1^{\bar{\mu}_1}(\mathbf{p}+\mathbf{r}_1)\,ds + \int_{\Gamma_2} \mathbf{v}^t v_2\mathbf{A}_2^{\bar{\mu}_2}(\mathbf{p}+\mathbf{r}_2)\,ds \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.49}$$

Integrating $\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{r}_{1,\xi_1} + \mathbf{A}_2\mathbf{r}_{2,\xi_2})\,d\xi$ by parts, (3.49) becomes

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{p}_{,\xi_1} + \mathbf{A}_2\mathbf{p}_{,\xi_2})\,d\xi - \iint_\Delta \mathbf{v}^t_{,\xi_1}\mathbf{A}_1\mathbf{r}_1 + \mathbf{v}^t_{,\xi_2}\mathbf{A}_2\mathbf{r}_2\,d\xi = \int_{\Gamma_1} \mathbf{v}^t v_1(\mathbf{A}_1^{\bar{\mu}_1}\mathbf{p} - \mathbf{A}_1^{\mu_1}\mathbf{r}_1)\,ds + \int_{\Gamma_2} \mathbf{v}^t v_2(\mathbf{A}_2^{\bar{\mu}_2}\mathbf{p} - \mathbf{A}_2^{\mu_2}\mathbf{r}_2)\,ds \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.50}$$

Since $\mathbf{r}_i(\tau,\xi_i) = L_{p+1}(\xi_i)\hat{\mathbf{c}}_i(\tau) - L_p(\xi_i)\hat{\mathbf{d}}_i(\tau)$, by the orthogonality of Legendre polynomials we have

$$\iint_\Delta \mathbf{v}^t_{,\xi_1}\mathbf{A}_1\mathbf{r}_1 + \mathbf{v}^t_{,\xi_2}\mathbf{A}_2\mathbf{r}_2\,d\xi = 0 \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.51}$$

Using (3.38c), (3.38e), $L_p(0) = (-1)^p$ and $L_p(1) = 1$, we further show that

$$\mathbf{A}_i^+\mathbf{r}_i(\tau,1) = \mathbf{A}_i^+(L_{p+1}(1)\hat{\mathbf{c}}_i(\tau) - L_p(1)\hat{\mathbf{d}}_i(\tau)) = \mathbf{A}_i^+\hat{\mathbf{c}}_i(\tau) - \mathbf{A}_i^+\hat{\mathbf{d}}_i(\tau) = 0, \tag{3.52a}$$

$$\mathbf{A}_i^-\mathbf{r}_i(\tau,0) = \mathbf{A}_i^-(L_{p+1}(0)\hat{\mathbf{c}}_i(\tau) - L_p(0)\hat{\mathbf{d}}_i(\tau)) = (-1)^p(\mathbf{A}_i^-\hat{\mathbf{c}}_i(\tau) + \mathbf{A}_i^-\hat{\mathbf{d}}_i(\tau)) = 0. \tag{3.52b}$$

Thus, we have established that

$$\mathbf{A}_i^{\mu_i}\mathbf{r}_i|_{\Gamma_i} = 0, \quad i = 1, 2. \tag{3.53}$$

Combining (3.51) and (3.53) with the orthogonality condition (3.50) leads to

$$\iint_\Delta \mathbf{v}^t(\mathbf{A}_1\mathbf{p}_{,\xi_1} + \mathbf{A}_2\mathbf{p}_{,\xi_2})\,d\xi = \int_\Gamma \mathbf{v}^t(v_1\mathbf{A}_1^{\bar{\mu}_1} + v_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{p}\,ds \quad \forall\ \mathbf{v} \in \mathscr{P}_p. \tag{3.54}$$

By Lemma 3.3, $\mathbf{p} = 0$ on Δ, which, when combined with (3.48), yields

$$\mathbf{q}_{p+1}(\tau,\xi) = \mathbf{r}_1(\tau,\xi_1) + \mathbf{r}_2(\tau,\xi_2). \tag{3.55}$$

This completes the proof. □

Corollary 3.1. *Under the conditions of Theorem 3.1 with both $\mathbf{A}_1$ and $\mathbf{A}_2$ invertible matrices, then the local DG error can be written as*

$$\mathbf{e}(t,h\xi) = h^{p+1}\left[\sum_{j=1}^{2}(\mathbf{M}_jR^+_{p+1}(\xi_j) + (\mathbf{M}_j - \mathbf{I})R^-_{p+1}(\xi_j))\mathbf{d}_j\right] + \mathcal{O}(h^{p+2}), \tag{3.56}$$

where

$$\mathbf{M}_j = \frac{1}{2}(\mathbf{I} + sgn(\mathbf{A}_j)), \quad j = 1, 2. \tag{3.57}$$

Moreover, if, for instance, only $\mathbf{A}_1$ is invertible, then the error can be written as

$$\mathbf{e}(t,h\xi) = h^{p+1}[(\mathbf{M}_1R^+_{p+1}(\xi_1) + (\mathbf{M}_1 - \mathbf{I})R^-_{p+1}(\xi_1))\mathbf{d}_1 + L_{p+1}(\xi_2)\mathbf{c}_2(t) + L_p(\xi_2)\mathbf{d}_2(t)] + \mathcal{O}(h^{p+2}), \tag{3.58}$$

where $\mathbf{c}_2$ and $\mathbf{d}_2$ satisfy (3.33b).

Proof. We prove the theorem when $\mathbf{A}_1$ and $\mathbf{A}_2$ are invertible. The proof for the other cases is similar and we will be omitted.

Combining (3.33b) we obtain

$$\mathbf{A}_i\mathbf{c}_i = (\mathbf{A}_i^+ - \mathbf{A}_i^-)\mathbf{d}_i, \quad i = 1, 2, \tag{3.59}$$

which can be written as

$$\mathbf{c}_i = \mathbf{A}_i^{-1}(\mathbf{A}_i^+ - \mathbf{A}_i^-)\mathbf{d}_i, \quad i = 1, 2. \tag{3.60}$$

Using $\mathbf{A}_i = \mathbf{P}_i\Lambda_i\mathbf{P}_i^t$ yields

$$\mathbf{c}_i = (\mathbf{P}_i\Lambda_i^{-1}\mathbf{P}_i^t\mathbf{P}_i|\Lambda_i|\mathbf{P}_i^t)\mathbf{d}_i = \mathbf{P}_i sgn(\Lambda)\mathbf{P}_i^t\mathbf{d}_i = sgn(\mathbf{A}_i)\mathbf{d}_i, \quad i = 1, 2. \tag{3.61}$$

Substituting the previous expression into (3.33a) we obtain

$$\mathbf{e}(t,h\xi) = h^{p+1}\sum_{j=1}^{2}(L_{p+1}(\xi_j)sgn(\mathbf{A}_j)\mathbf{d}_j(t) - L_p(\xi_j)\mathbf{d}_j(t)) + \mathcal{O}(h^{p+2}), \tag{3.62}$$

which can be written as

$$\mathbf{e}(t,h\xi) = h^{p+1}\sum_{j=1}^{2}\left((L_{p+1}(\xi_j) - L_p(\xi_j))\left[\frac{I + sgn(\mathbf{A}_j)}{2}\right] + (L_{p+1}(\xi_j) + L_p(\xi_j))\left[\frac{-I + sgn(\mathbf{A}_j)}{2}\right]\right)\mathbf{d}_j + \mathcal{O}(h^{p+2}). \quad (3.63)$$

This completes the proof. □

4. Superconvergence and *a posteriori* error analysis

In this section we investigate pointwise superconvergence for DG solutions and describe procedures to compute asymptotically correct *a posteriori* DG error estimates under mesh refinement.

4.1. Superconvergence

In order for the DG solution $\mathbf{u}_h$ to be $\mathcal{O}(h^{p+2})$-superconvergent at few points in element ω, the leading error term shown in Theorem 3.1 has to be zero at these points. This pointwise superconvergence happens only for special hyperbolic problems as shown in the following theorem:

Theorem 4.1. *Under the conditions of Theorem 3.1 we let $\bar{\xi}_j^s$, $j = 1, \cdots, p+1$, denote the roots of $R_{p+1}^s(\xi)$, $s = +, -$ shifted to $[0,1]$. Thus,*

1. If $\mathbf{z}$ is a unit vector in the union of the spaces $\mathcal{R}(\mathbf{A}_1^s)\bigcap\mathcal{R}(\mathbf{A}_2^\sigma)$, $s,\sigma = +,-$, then the projection $\mathbf{z}^t\mathbf{e}(t,\mathbf{x})$ of the local error onto *span*$\{\mathbf{z}\}$ is $\mathcal{O}(h^{p+2})$ superconvergent at the points $(t,\bar{\mathbf{x}}) = (t, h\bar{\xi}_k^s, h\bar{\xi}_l^\sigma)$, $k,l = 1,2,\ldots,p+1$, for $t = \mathcal{O}(1)$, i.e.,
$$\mathbf{z}^t\mathbf{e}(t,h\bar{\xi}_k^s,h\bar{\xi}_l^\sigma) = \mathcal{O}(h^{p+2}). \quad (4.1)$$
2. Moreover, If $\gamma_i(\alpha) = \{\mathbf{x}\in(0,h)^2 : x_i = \alpha\}, 0 \leqslant \alpha \leqslant h$ and if $\mathbf{v}\in\mathcal{P}_{p-1}$ is a unit vector with respect to the C^∞ norm, then for $\alpha = h\bar{\xi}_k^s$, $k = 1,\ldots,p+1$, we have the superconvergence of the following error averages
$$\frac{1}{h}\int_{\gamma_i(h\bar{\xi}_l^s)}\mathbf{v}^t\mathbf{A}_i^s\mathbf{e}\,ds = \mathcal{O}(h^{p+2}), \quad i = 1,2,\ s = +,- \quad (4.2)$$
and
$$\frac{1}{h}\int_{\gamma_i^s}\mathbf{v}^t(\mathbf{A}_i^{\mu_i}\mathbf{e} + \mathbf{A}_i^{\bar{\mu}_i}\mathbf{e}^-)\,ds = \mathcal{O}(h^{p+2}), \quad i = 1,2,\ s = +,-. \quad (4.3)$$

Proof. We prove (4.1) by assuming that there exists a unit vector $\mathbf{z}\in\mathcal{R}(\mathbf{A}_i^+)$ for $i=1$ and $i=2$, i.e., there exists $\mathbf{v}_i$ such that $\mathbf{A}_i^+\mathbf{v}_i = \mathbf{z}$, $i = 1,2$.

Left pre-multiplying (3.33a) by $\mathbf{z}^t$ and evaluating the resulting function at the points $(t,h\bar{\xi}_k^+,h\bar{\xi}_l^+), k,l = 1,\ldots,p+1$ obtain

$$\mathbf{z}^t\mathbf{e}(t,h\bar{\xi}_k^+,h\bar{\xi}_l^+) = h^{p+1}(L_{p+1}(\bar{\xi}_k^+)\mathbf{z}^t\mathbf{c}_1 - L_p(\bar{\xi}_k^+)\mathbf{z}^t\mathbf{d}_1) + h^{p+1}(L_{p+1}(\bar{\xi}_l^+)\mathbf{z}^t\mathbf{c}_2 - L_p(\bar{\xi}_l^+)\mathbf{z}^t\mathbf{d}_2) + \mathcal{O}(h^{p+2}). \quad (4.4)$$

Applying (3.33b) yields

$$\mathbf{z}^t\mathbf{d}_i = \mathbf{v}_i^t\mathbf{A}_i^+\mathbf{d}_i = \mathbf{v}_i^t\mathbf{A}_i^+\mathbf{c}_i = \mathbf{z}^t\mathbf{c}_i. \quad (4.5)$$

Combining (4.4) and (4.5) we prove that

$$\mathbf{z}^t\mathbf{e}(t,h\bar{\xi}_k^+,h\bar{\xi}_l^+) = \mathcal{O}(h^{p+2}). \quad (4.6)$$

Following the same line of reasoning we establish (4.1) for all other cases.

Let $\mathbf{v}\in\mathcal{P}_{p-1}$ be a unit vector in $[C^\infty]^m$ norm and apply the orthogonality of Legendre polynomials and the relations (3.33a) to obtain

$$\begin{aligned}\frac{1}{h}\int_{\gamma_1(h\bar{\xi}_k^+)}\mathbf{v}^t\mathbf{A}_1^+\mathbf{e}\,ds &= \frac{1}{h}\int_0^h\mathbf{v}^t(h\bar{\xi}_k^+,x_2)\mathbf{A}_1^+\mathbf{e}(t,h\bar{\xi}_k^+,x_2)\,dx_2\\ &= \int_0^1\mathbf{v}^t(h\bar{\xi}_k^+,h\xi_2)\mathbf{A}_1^+\mathbf{e}(t,h\bar{\xi}_k^+,h\xi_2)\,d\xi_2\\ &= h^{p+1}\int_0^1\mathbf{v}^t\mathbf{A}_1^+(L_{p+1}(\bar{\xi}_k^+)\mathbf{c}_1 - L_p(\bar{\xi}_k^+)\mathbf{d}_1\\ &\quad + L_{p+1}(\xi_2)\mathbf{c}_2 - L_p(\xi_2)\mathbf{d}_2)\,d\xi_2 + \mathcal{O}(h^{p+2})\\ &= h^{p+1}\int_0^1\mathbf{v}^t(L_{p+1}(\bar{\xi}_k^+) - L_p(\bar{\xi}_k^+))\mathbf{A}_1^+\mathbf{d}_1\,d\xi_2 + \mathcal{O}(h^{p+2})\\ &= \mathcal{O}(h^{p+2}).\end{aligned}$$

The estimate (4.2) holds on $\gamma_i(h\bar{\xi}_k^\pm)$, $i = 1,2$ for $\mathbf{A}_i^\pm$.

By virtue of (3.4) we note that on γ_1^+

$$\begin{aligned}\mathbf{e}^-(t,\mathbf{x}) &= \mathbf{u}_B(t,\mathbf{x}) - \pi_1\mathbf{u}_B(t,\mathbf{x})\\ &= h^{p+1}\left[L_{p+1}\left(\frac{x_2}{h}\right)\mathbf{c}_2 + L_p\left(\frac{x_2}{h}\right)\mathbf{d}_2\right] + \mathcal{O}(h^{p+2}).\end{aligned} \quad (4.7)$$

Since 1 and 0, respectively, are shifted roots of R_{p+1}^+ and R_{p+1}^-, applying (4.2) we will prove $\mathcal{O}(h^{p+2})$ superconvergence of the flux for $i = 1$.

$$\begin{aligned}&\frac{1}{h}\int_{\gamma_1^+}\mathbf{v}^t(\mathbf{A}_1^+\mathbf{e} + \mathbf{A}_1^-\mathbf{e}^-)\,ds\\ &= \frac{1}{h}\left[\int_{\gamma_1^+}\mathbf{v}^t\mathbf{A}_1^+\mathbf{e}\,ds + h^{p+1}\int_{\gamma_1^+}\mathbf{v}^t\left[L_{p+1}\left(\frac{x_2}{h}\right)\mathbf{A}_1^-\mathbf{c}_2 + L_p\left(\frac{x_2}{h}\right)\mathbf{A}_1^-\mathbf{d}_2\right]ds\right]\\ &\quad + \mathcal{O}(h^{p+2})\end{aligned} \quad (4.8)$$

Now, combining (4.8) and (4.2) yields (4.3). We presented the proof for $i = 1$ with $\alpha = h$. Other cases can be treated using the same line of reasoning and are omitted. □

4.2. A posteriori error estimation

In this section we present an *a posteriori* error estimation procedure which consists of computing asymptotically exact local and global error estimates of the DG error. In Theorem 3.1 we showed that the local discretization error for the DG method on a physical element $\omega = (0,h)^2$ can be written as

$$\begin{aligned}\mathbf{e}(t,h\xi) &= \mathbf{u}(t,h\xi) - \mathbf{u}_h(t,h\xi)\\ &= h^{p+1}\sum_{j=1}^{2}(L_{p+1}(\xi_j)\mathbf{c}_j(t) - L_p(\xi_j)\mathbf{d}_j(t)) + \mathcal{O}(h^{p+2}),\end{aligned} \quad (4.9a)$$

where $\mathbf{c}_i(t), \mathbf{d}_i(t), i = 1,2$ satisfy

$$\mathbf{A}_i^+\mathbf{c}_i(t) - \mathbf{A}_i^+\mathbf{d}_i(t) = 0, \qquad \mathbf{A}_i^-\mathbf{c}_i(t) + \mathbf{A}_i^-\mathbf{d}_i(t) = 0, \quad i = 1,2. \quad (4.9b)$$

In this manuscript we consider problems where the matrices $\mathbf{A}_i$, $i = 1,2$ may be singular and satisfy assumptions of Lemma 3.3.

Let us subtract Eq. (4.9b) and solve for $\mathbf{d}_i$ in terms of $\mathbf{c}_i$ to write

$$\mathbf{d}_i = \mathbf{A}_i^\dagger(\mathbf{A}_i^+ - \mathbf{A}_i^-)\mathbf{c}_i + \mathbf{d}_i^{\maltese} = sgn(\mathbf{A}_i)\mathbf{c}_i^\perp + \mathbf{d}_i^{\maltese}. \quad (4.10a)$$

where $\mathbf{A}_i^\dagger$ denotes the pseudoinverse of $\mathbf{A}_i$, $1 \leqslant i \leqslant d$. Moreover, from the direct sum $\mathbb{R}^m = \mathcal{N}(\mathbf{A}_i) \oplus \mathcal{N}(\mathbf{A}_i)^\perp$ we split $\mathbf{c}_i$ as

$$\mathbf{c}_i = \mathbf{c}_i^\perp + \mathbf{c}_i^{\maltese}, \quad (4.10b)$$

where $\mathbf{c}_i^{\maltese}$, $\mathbf{d}_i^{\maltese} \in \mathcal{N}(\mathbf{A}_i)$ and $\mathbf{c}_i^\perp$, $\mathbf{d}_i^\perp \in \mathcal{N}(\mathbf{A}_i)^\perp$.

Hence, the leading term of the spatial discretization error (4.9a) can be split into two parts as

$$\mathbf{e} = \mathbf{e}^{\perp} + \mathbf{e}^{\maltese} + \mathcal{O}(h^{p+2}), \tag{4.11a}$$

where

$$\mathbf{e}^{\perp}(t, h\xi) = h^{p+1} \sum_{j=1}^{2} [L_{p+1}(\xi_j)\mathbf{I} - L_p(\xi_j) sgn(\mathbf{A}_j)] \mathbf{c}_j^{\perp}(t), \tag{4.11b}$$

$$\mathbf{e}^{\maltese}(t, h\xi) = h^{p+1} \sum_{j=1}^{2} (L_{p+1}(\xi_j)\mathbf{c}_j^{\maltese}(t) - L_p(\xi_j)\mathbf{d}_j^{\maltese}(t)). \tag{4.11c}$$

We note that for invertible matrices $\mathbf{A}_i$, $i = 1, 2$, the error component $\mathbf{e}^{\maltese}(t, \mathbf{x}) = 0$.

Next, we develop an *a posteriori* error estimation procedure for estimating both $\mathbf{e}^{\perp}$ and $\mathbf{e}^{\maltese}$ (if needed). We end the section by proving that, for smooth solutions, our local error estimates converge to the true error under mesh refinement. Up to this point we are not able to prove the asymptotic exactness of our global *a posteriori* error estimates. However, computational results for several hyperbolic systems shown in Section 5 suggest that our global *a posteriori* error estimates are asymptotically exact under mesh refinement for smooth solutions.

The *a posteriori* error estimation procedure to compute estimates for $\mathbf{e}^{\perp}$ consists of determining

$$\mathbf{E}^{\perp}(t, h\xi) = \sum_{j=1}^{2} [L_{p+1}(\xi_j)\mathbf{I} - L_p(\xi_j) sgn(\mathbf{A}_j)] \gamma_j^{\perp}(t)), \quad \gamma_j^{\perp} \in \mathcal{N}(\mathbf{A}_j)^{\perp}, \tag{4.12a}$$

such that

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right) \mathbf{w}^t [\mathbf{u}_{h,t} + \mathbf{A}_1(\mathbf{u}_h + \mathbf{E}^{\perp})_{,x_1} + \mathbf{A}_2(\mathbf{u}_h + \mathbf{E}^{\perp})_{,x_2} - \mathbf{f}]\, d\mathbf{x} = 0, \quad \mathbf{w} \in \mathcal{N}(\mathbf{A}_i)^{\perp}, \quad i = 1,\ 2. \tag{4.12b}$$

Note that $\mathsf{P}_i = \mathbf{A}_i^{\dagger}\mathbf{A}_i$ is symmetric and projects any vector in $\mathbb{R}^m$ into $\mathcal{N}(\mathbf{A}_i)^{\perp}$. Therefore, the columns of P_i span $\mathcal{N}(\mathbf{A}_i)^{\perp}$. Testing against all columns of P_i, we can replace (4.12b) by the system

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right) \mathsf{P}_i [\mathbf{u}_{h,t} + \mathbf{A}_1(\mathbf{u}_h + \mathbf{E}^{\perp})_{,x_1} + \mathbf{A}_2(\mathbf{u}_h + \mathbf{E}^{\perp})_{,x_2} - \mathbf{f}]\, d\mathbf{x} = 0, \quad i = 1,\ 2, \tag{4.13}$$

which can be reduced to

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right) L'_{p+1}\left(\frac{x_i}{h}\right) d\mathbf{x}\ \mathbf{A}_i \gamma_i^{\perp} = \mathfrak{r}_{p,i}^{\perp}, \tag{4.14a}$$

where $\mathfrak{r}_{p,i}^{\perp}$ is the projection of the residual defined as

$$\mathfrak{r}_{p,i}^{\perp} = \iint_{\omega} L_p\left(\frac{x_i}{h}\right) \mathsf{P}_i (\mathbf{f} - \mathbf{u}_{h,t} - \mathbf{A}_1\mathbf{u}_{h,x_1} - \mathbf{A}_2\mathbf{u}_{h,x_2})\, d\mathbf{x}. \tag{4.14b}$$

This can be reduced further to obtain

$$2h\ \mathbf{A}_i \gamma_i^{\perp} = \mathfrak{r}_{p,i}^{\perp}, \quad i = 1, 2. \tag{4.15}$$

Since $\mathfrak{r}_{p,i}^{\perp} \in \mathcal{N}(\mathbf{A}_i)^{\perp}$, we can solve (4.15) to find the unique solution $\gamma_i^{\perp} \in \mathcal{N}(\mathbf{A}_i)^{\perp}$,

$$\gamma_i^{\perp} = \frac{1}{2h} \mathbf{A}_i^{\dagger} \mathfrak{r}_{p,i}^{\perp}, \quad i = 1, 2. \tag{4.16}$$

Now, we turn to estimating the error component $\mathbf{e}^{\maltese}$ lying in the following polynomial space:

$$\mathscr{E}^p = \left\{ \mathbf{v}, \mathbf{v}(\mathbf{x}) = \sum_{i=1}^{2} \left(L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_i}{h}\right)\mathbf{b}_i \right), \mathbf{a}_i, \mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i) \right\}, \tag{4.17}$$

which contains nonzero elements only if at least $\mathbf{A}_1$ or $\mathbf{A}_2$ is singular.

By Lemma 3.1, $\mathbf{u}_B - \mathbf{u}_h$ on $\partial\omega$ and $(\mathbf{u}_0 - \mathbf{u}_h)(0, \mathbf{x})$ on ω satisfy

$$(\mathbf{u}_B - \mathbf{u}_h^-)(t, h\xi) = h^{p+1}(L_{p+1}(\xi_i)\mathbf{c}_i(t) - L_p(\xi_i)\mathbf{d}_i(t)) + \mathcal{O}(h^{p+2}) \quad \forall\, \mathbf{x} \in \gamma_{i'}, \quad i = 1, 2 \tag{4.18}$$

$$(\mathbf{u}_0 - \mathbf{u}_h)(0, h\xi) = \sum_{j=1}^{2} h^{p+1}(L_{p+1}(\xi_j)\mathbf{c}_j(0) - L_p(\xi_j)\mathbf{d}_j(0)) + \mathcal{O}(h^{p+2}) \quad \forall\, \mathbf{x} \in \omega, \tag{4.19}$$

where $\mathbf{c}_i(t), \mathbf{d}_i(t), i = 1, 2$, satisfy

$$\mathbf{A}_i^+ \mathbf{c}_i(t) - \mathbf{A}_i^+ \mathbf{d}_i(t) = 0, \qquad \mathbf{A}_i^- \mathbf{c}_i(t) + \mathbf{A}_i^- \mathbf{d}_i(t) = 0, \quad i = 1, 2. \tag{4.20}$$

Therefore, we can define $\mathbf{E}^{\maltese}(0, \mathbf{x})$ on ω and $\mathbf{E}^-(t, \mathbf{x})$ on $\partial\omega$ by

$$\mathbf{E}^{\maltese}(0, \mathbf{x}) = \mathbf{e}^{\maltese}(0, \mathbf{x}) \quad \forall\, \mathbf{x} \in \omega \tag{4.21a}$$

and

$$\mathbf{E}^-(t, \mathbf{x}) = h^{p+1}(L_{p+1}(\xi_i)\mathbf{c}_i(t) - L_p(\xi_i)\mathbf{d}_i(t)) \quad \forall\, \mathbf{x} \in \gamma_{i'}, \quad i = 1, 2, \tag{4.21b}$$

where i' denotes the dual of i as defined in (3.15).

Now, let us approximate $\mathbf{e}^{\maltese}$ by determining

$$\mathbf{E}^{\maltese}(t, h\xi) = \sum_{j=1}^{2} L_{p+1}(\xi_j)\gamma_j^{\maltese} - L_p(\xi_j)\delta_j^{\maltese}, \tag{4.22a}$$

such that

$$\iint_{\omega} \mathbf{v}^t [(\mathbf{u}_h + \mathbf{E}^{\maltese})_{,t} + \mathbf{A}_1\mathbf{u}_{h,x_1} + \mathbf{A}_2\mathbf{u}_{h,x_2} - \mathbf{f}]\, d\mathbf{x} + \int_{\partial\omega} \mathbf{v}^t (\nu_1 \mathbf{A}_1^{\bar{\mu}_1} + \nu_2 \mathbf{A}_2^{\bar{\mu}_2})(\mathbf{u}_h^- + \mathbf{E}^- - \mathbf{u}_h - \mathbf{E}^{\perp} - \mathbf{E}^{\maltese})\, ds = 0 \quad \forall\, \mathbf{v} \in \mathscr{E}^p. \tag{4.22b}$$

Since $(\mathbf{I} - \mathsf{P}_i)$ projects any vector $\mathbf{w} \in \mathbb{R}^m$ into $\mathcal{N}(\mathbf{A}_i)$, the columns of $(\mathbf{I} - \mathsf{P}_i)$ span $\mathcal{N}(\mathbf{A}_i)$. Hence, the columns of $L_{p+1}(\xi_i)(\mathbf{I} - \mathsf{P}_i)$ and $L_p(\xi_i)(\mathbf{I} - \mathsf{P}_i)$ span $\mathscr{E}^p$.

Applying the projection $(\mathbf{I} - \mathsf{P}_i)$ to (4.22b) yields the following system:

$$\iint_{\omega} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)[(\mathbf{u}_h + \mathbf{E}^{\maltese})_{,t} + \mathbf{A}_1\mathbf{u}_{h,x_1} + \mathbf{A}_2\mathbf{u}_{h,x_2} - \mathbf{f}]\, d\mathbf{x} + \int_{\partial\omega} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)(\nu_1 \mathbf{A}_1^{\bar{\mu}_1} + \nu_2 \mathbf{A}_2^{\bar{\mu}_2})(\mathbf{u}_h^- + \mathbf{E}^- - \mathbf{u}_h - \mathbf{E}^{\perp} - \mathbf{E}^{\maltese})\, ds = 0, \quad m = p, p+1, \quad i = 1, 2. \tag{4.23}$$

Using $(\mathbf{I} - \mathsf{P}_i)\mathbf{A}_i^{\bar{\mu}_i} = 0$, (4.23) can be written as

$$\iint_{\omega} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)\mathbf{E}_{,t}^{\maltese}\, d\mathbf{x} - \int_{\gamma_{i'}} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)(\nu_{i'}\mathbf{A}_{i'}^{\bar{\mu}_{i'}})\mathbf{E}^{\maltese}\, ds = \mathfrak{r}_{m,i}^{\maltese}, \tag{4.24a}$$

where $\mathfrak{r}_{m,i}^{\maltese}$ is the projection of the residual given by

$$\mathfrak{r}_{m,i}^{\maltese} = -\iint_{\omega} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)(\mathbf{u}_{h,t} + \mathbf{A}_1\mathbf{u}_{h,x_1} + \mathbf{A}_2\mathbf{u}_{h,x_2} - \mathbf{f})\, d\mathbf{x} - \int_{\gamma_{i'}} L_m\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)(\nu_{i'}\mathbf{A}_{i'}^{\bar{\mu}_{i'}})(\mathbf{u}_h^- + \mathbf{E}^- - \mathbf{u}_h - \mathbf{E}^{\perp})\, ds \tag{4.24b}$$

with i' denoting the dual of i defined in (3.15).

For $m = p + 1$, (4.24a) can be reduced to

$$\iint_{\omega} L_{p+1}^2\left(\frac{x_i}{h}\right) d\mathbf{x}\, \frac{d}{dt}\gamma_i^{\maltese} - \int_{\gamma_{i'}} L_{p+1}^2\left(\frac{x_i}{h}\right)(\mathbf{I} - \mathsf{P}_i)(\nu_{i'}\mathbf{A}_{i'}^{\bar{\mu}_{i'}})\gamma_i^{\maltese}\, ds = \mathfrak{r}_{p+1,i}^{\maltese}, \tag{4.25}$$

which is equal to

$$h^2 \frac{d}{dt}\gamma_i^{\maltese} + h(\mathbf{I} - \mathsf{P}_i)(\mathbf{A}_{i'}^+ - \mathbf{A}_{i'}^-)\gamma_i^{\maltese} = (2p+3)\mathfrak{r}_{p+1,i}^{\maltese}. \tag{4.26a}$$

For $m = p$, we get similarly

$$h^2 \frac{d}{dt}\delta_i^{\maltese} + h(\mathbf{I} - \mathrm{P}_i)(\mathbf{A}_{i'}^{+} - \mathbf{A}_{i'}^{-})\delta_i^{\maltese} = (2p+1)\mathfrak{r}_{p,i}^{\maltese} \tag{4.26b}$$

subject to the initial conditions

$$\gamma_i^{\maltese}(0) = \mathbf{c}_i(0), \quad \delta_i^{\maltese}(0) = \mathbf{d}_i(0). \tag{4.26c}$$

We are ready to state and establish the convergence of the local error estimate for $\mathbf{e}^{\perp}$ and $\mathbf{e}^{\maltese}$.

Theorem 4.2. *Under the assumptions of Theorem 3.1 with $p \geqslant 1$, let us consider the error estimate*

$$\mathbf{E}^{\perp}(t, h\xi) = \sum_{j=1}^{2}[L_{p+1}(\xi_j) - L_p(\xi_j)sgn(\mathbf{A}_j)]\frac{1}{2h}\mathbf{A}_j^{\dagger}\mathfrak{r}_{p,j}^{\perp}, \tag{4.27}$$

where $\mathfrak{r}_{p,j}$, $j = 1, 2$ are defined in (4.14b). Then, at $t = \mathcal{O}(1)$,

$$\mathbf{e}^{\perp}(t, \mathbf{x}) = \mathbf{E}^{\perp}(t, \mathbf{x}) + \mathcal{O}(h^{p+2}) \quad \forall\, \mathbf{x} \in \omega. \tag{4.28}$$

Proof. Since the true solution $\mathbf{u}$ satisfies Eq. (2.1), we have

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right)\mathrm{P}_i[\mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,x_1} + \mathbf{A}_2\mathbf{u}_{,x_2} - \mathbf{f}]\,d\mathbf{x} = 0, \quad i = 1, 2. \tag{4.29}$$

Subtracting (4.13) from (4.29) yields

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right)\mathrm{P}_i[\mathbf{e}_{,t} + \mathbf{A}_1(\mathbf{e} - \mathbf{E}^{\perp})_{,x_1} + \mathbf{A}_2(\mathbf{e} - \mathbf{E}^{\perp})_{,x_2}]\,d\mathbf{x} = 0, \quad i = 1, 2. \tag{4.30}$$

Applying (4.11a) and $\mathbf{A}_i\mathbf{e}_{,x_i}^{\maltese} = 0$, $i = 1, 2$, (4.30) becomes

$$\iint_{\omega} L_p\left(\frac{x_i}{h}\right)\mathrm{P}_i[\mathbf{e}_{,t} + \mathbf{A}_1(\mathbf{e}^{\perp} - \mathbf{E}^{\perp})_{,x_1} + \mathbf{A}_2(\mathbf{e}^{\perp} - \mathbf{E}^{\perp})_{,x_2}]\,d\mathbf{x} = 0, \quad i = 1, 2. \tag{4.31}$$

Applying the linear transformations $t = T\tau$, $T > 0$, and $\mathbf{x} = h\xi$, (4.31) becomes

$$\iint_{\Delta} L_p(\xi_i)\mathrm{P}_i\left[\frac{h}{T}\hat{\mathbf{e}}_{,\tau} + \mathbf{A}_1(\hat{\mathbf{e}}^{\perp} - \hat{\mathbf{E}}^{\perp})_{,\xi_1} + \mathbf{A}_2(\hat{\mathbf{e}}^{\perp} - \hat{\mathbf{E}}^{\perp})_{,\xi_2}\right]d\xi = 0. \tag{4.32}$$

where $\hat{\mathbf{e}}(\tau, \xi) = \mathbf{e}(\tau T, h\xi)$. By substituting the definitions of $\mathbf{e}^{\perp}$ (4.11b) and $\mathbf{E}^{\perp}$ (4.12a) into (4.32) we get

$$\iint_{\Delta} \mathrm{P}_i\left[\frac{h^{p+2}}{T}L_p^2(\xi_i)sgn(\mathbf{A}_i)\hat{\mathbf{c}}_{,\tau} + L_p(\xi_i)L'_{p+1}(\xi_i)\mathbf{A}_i(h^{p+1}\hat{\mathbf{c}}_i^{\perp} - \hat{\gamma}_i^{\perp})\right]d\xi = \mathcal{O}(h^{p+2}), \tag{4.33}$$

where we used the orthogonality properties of Legendre polynomials. This can be further simplified to

$$\frac{h^{p+2}}{T(2p+1)}sgn(\mathbf{A}_i)\hat{\mathbf{c}}_{,\tau} + 2\mathbf{A}_i(h^{p+1}\hat{\mathbf{c}}_i^{\perp} - \hat{\gamma}_i^{\perp}) = \mathcal{O}(h^{p+2}). \tag{4.34}$$

Thus, at $T = \mathcal{O}(1)$ we have

$$2\mathbf{A}_i(h^{p+1}\mathbf{c}_i^{\perp} - \gamma_i^{\perp}) = \mathcal{O}(h^{p+2}). \tag{4.35}$$

Since $\mathbf{c}_i^{\perp}$, $\gamma_i^{\perp} \in \mathcal{N}(\mathbf{A}_i)^{\perp}$, it follows that:

$$\gamma_i^{\perp}(t) = \mathbf{c}_i^{\perp}(t) + \mathcal{O}(h^{p+2}), \tag{4.36}$$

which establishes (4.28). □

Next, we will state and prove a technical lemma before stating and proving our last theorem.

Lemma 4.1. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be symmetric matrices that satisfy the assumptions of Lemma 3.3 and $\mathbf{q} \in \mathcal{E}^p$. If $\mathbf{q}$ satisfies the orthogonality condition on the reference element*

$$\iint_{\Delta} \mathbf{v}^t[\mathbf{A}_1\mathbf{q}_{,\xi_1} + \mathbf{A}_2\mathbf{q}_{,\xi_2}]\,d\xi - \int_{\Gamma} \mathbf{v}^t(\nu_1\mathbf{A}_1^{\mu_1} + \nu_2\mathbf{A}_2^{\mu_2})\mathbf{q}\,ds = 0 \quad \forall\, \mathbf{v} \in \mathcal{E}^p, \tag{4.37}$$

then $\mathbf{q} = 0$.

Proof. First we integrate Eq. (4.37) by parts to write

$$-\iint_{\Delta}(\mathbf{v}_{,\xi_1}^t\mathbf{A}_1 + \mathbf{v}_{,\xi_2}^t\mathbf{A}_2)\mathbf{q}\,d\xi + \int_{\Gamma}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\mu_1} + \nu_2\mathbf{A}_2^{\mu_2})\mathbf{q}\,ds = 0. \tag{4.38}$$

Adding (4.37) and (4.38) and setting $\mathbf{v} = \mathbf{q}$, the integral on Δ vanishes because of the symmetry of $\mathbf{A}_1$ and $\mathbf{A}_2$, and we get

$$\int_{\Gamma_1}\mathbf{q}^t(\mathbf{A}_1^{+} - \mathbf{A}_1^{-})\mathbf{q}\,ds + \int_{\Gamma_2}\mathbf{q}^t(\mathbf{A}_2^{+} - \mathbf{A}_2^{-})\mathbf{q}\,ds = 0. \tag{4.39}$$

Since $\mathbf{q} \in \mathcal{E}^p$, there are $\mathbf{a}_i, \mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i)$ for $i = 1, 2$, such that

$$\mathbf{q}(\xi) = \sum_{i=1}^{2} L_{p+1}(\xi_i)\mathbf{a}_i - L_p(\xi_i)\mathbf{b}_i. \tag{4.40}$$

Combining $\mathcal{N}(\mathbf{A}_i) = \mathcal{N}(\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$, $(\mathbf{A}_i^{+} - \mathbf{A}_i^{-})\mathbf{a}_i = (\mathbf{A}_i^{+} - \mathbf{A}_i^{-})\mathbf{b}_i = 0$ for $i = 1, 2$, and (4.39) to obtain

$$\frac{1}{2p+3}\mathbf{a}_2^t(\mathbf{A}_1^{+} - \mathbf{A}_1^{-})\mathbf{a}_2 + \frac{1}{2p+1}\mathbf{b}_2^t(\mathbf{A}_1^{+} - \mathbf{A}_1^{-})\mathbf{b}_2 + \frac{1}{2p+3}\mathbf{a}_1^t(\mathbf{A}_2^{+} - \mathbf{A}_2^{-})\mathbf{a}_1 + \frac{1}{2p+1}\mathbf{b}_1^t(\mathbf{A}_2^{+} - \mathbf{A}_2^{-})\mathbf{b}_1 = 0. \tag{4.41}$$

Since $(\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$ is positive semidefinite, there exists a matrix $\mathbf{L}_i$ such that $\mathbf{L}_i^t\mathbf{L}_i = (\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$ for $i = 1, 2$ and (4.41) can be written as

$$\frac{1}{2p+3}\|\mathbf{L}_1\mathbf{a}_2\|_2^2 + \frac{1}{2p+1}\|\mathbf{L}_1\mathbf{b}_2\|_2^2 + \frac{1}{2p+3}\|\mathbf{L}_2\mathbf{a}_1\|_2^2 + \frac{1}{2p+1}\|\mathbf{L}_2\mathbf{b}_1\|_2^2 = 0. \tag{4.42}$$

This leads to

$$\mathbf{L}_1\mathbf{a}_2 = \mathbf{L}_1\mathbf{b}_2 = \mathbf{L}_2\mathbf{a}_1 = \mathbf{L}_2\mathbf{b}_1 = 0. \tag{4.43}$$

We premultiply $\mathbf{L}_1\mathbf{a}_2$ and $\mathbf{L}_1\mathbf{b}_2$ by $\mathbf{L}_1^t$ and $\mathbf{L}_2\mathbf{a}_1$ and $\mathbf{L}_2\mathbf{b}_1$ by $\mathbf{L}_2^t$ to show that

$$(\mathbf{A}_1^{+} - \mathbf{A}_1^{-})\mathbf{a}_2 = (\mathbf{A}_1^{+} - \mathbf{A}_1^{-})\mathbf{b}_2 = (\mathbf{A}_2^{+} - \mathbf{A}_2^{-})\mathbf{a}_1 = (\mathbf{A}_2^{+} - \mathbf{A}_2^{-})\mathbf{b}_1 = 0. \tag{4.44}$$

Combining Lemmas 3.2 and 3.3, which yield $\mathcal{N}(\mathbf{A}_1) \cap \mathcal{N}(\mathbf{A}_2) = \{0\}$, the definition of $\mathcal{E}^p$ and $\mathcal{N}(\mathbf{A}_i) = \mathcal{N}(\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$ to infer that $\mathbf{a}_2 = \mathbf{b}_2 = 0$ and $\mathbf{a}_1 = \mathbf{b}_1 = 0$. Thus, we establish Lemma 4.1. □

Now, we state and prove the convergence of the error estimate $\mathbf{E}^{\maltese}$ to $\mathbf{e}^{\maltese}$ under mesh refinement.

Theorem 4.3. *Under the assumptions of Theorem 3.1 with $p \geqslant 1$ we let*

$$\mathbf{E}^{\maltese}(t, h\xi) = \sum_{j=1}^{2}(L_{p+1}(\xi_j)\gamma_j^{\maltese}(t) - L_p(\xi_j)\delta_j^{\maltese}(t)), \tag{4.45}$$

where $\gamma_i^{\maltese}, \delta_i^{\maltese}$, $i = 1, 2$ are solutions of (4.26a) and (4.24b). Then, at $t = \mathcal{O}(1)$,

$$\mathbf{e}^{\maltese}(t, \mathbf{x}) = \mathbf{E}^{\maltese}(t, \mathbf{x}) + \mathcal{O}(h^{p+2}) \quad \forall\, \mathbf{x} \in \omega. \tag{4.46}$$

Proof. Since the true solution $\mathbf{u}$ is continuous and $\mathbf{u} = \mathbf{u}^{-}$ on $\partial\omega$, $\mathbf{u}$ satisfies

$$\iint_{\omega}\mathbf{v}^t[\mathbf{u}_{,t}+\mathbf{A}_1\mathbf{u}_{,x_1}+\mathbf{A}_2\mathbf{u}_{,x_2}-\mathbf{f}]\,d\mathbf{x} + \int_{\partial\omega}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})(\mathbf{u}^- - \mathbf{u})\,ds = 0 \quad \forall\,\mathbf{v}\in\mathscr{E}^p. \tag{4.47}$$

Subtracting (4.22b) from (4.47) gives

$$\iint_{\omega}\mathbf{v}^t[(\mathbf{e}-\mathbf{E}^{\maltese})_{,t}+\mathbf{A}_1\mathbf{e}_{,x_1}+\mathbf{A}_2\mathbf{e}_{,x_2}]\,d\mathbf{x} + \int_{\partial\omega}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})(\mathbf{e}^- - \mathbf{E}^- - \mathbf{e} + \mathbf{E}^{\perp} + \mathbf{E}^{\maltese})\,ds = 0. \tag{4.48}$$

Using the definition of $\mathbf{E}^{\perp}$ and $\mathbf{E}^{\maltese}$ and the properties of Legendre polynomials one can easily check that the following holds

$$\iint_{\omega}\mathbf{v}^t[\mathbf{E}^{\perp}_{,t}+\mathbf{A}_1(\mathbf{E}^{\perp}+\mathbf{E}^{\maltese})_{,x_1}+\mathbf{A}_2(\mathbf{E}^{\perp}+\mathbf{E}^{\maltese})_{,x_2}]\,d\mathbf{x} = 0 \quad \forall\,\mathbf{v}\in\mathscr{E}^p. \tag{4.49}$$

Subtracting (4.49) from (4.48) gives for $\epsilon = \mathbf{e} - \mathbf{E}^{\perp} - \mathbf{E}^{\maltese}$ and $\epsilon^- = \mathbf{e}^- - \mathbf{E}^-$

$$\iint_{\omega}\mathbf{v}^t[\epsilon_{,t}+\mathbf{A}_1\epsilon_{,x_1}+\mathbf{A}_2\epsilon_{,x_2}]\,d\mathbf{x} + \int_{\partial\omega}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})(\epsilon^- - \epsilon)\,ds = 0. \tag{4.50}$$

Applying the linear transformations $t = T\tau$, $T > 0$, and $\mathbf{x} = h\xi$, (4.50) becomes

$$\iint_{\Delta}\mathbf{v}^t\left[\frac{h}{T}\epsilon_{,\tau}+\mathbf{A}_1\epsilon_{,\xi_1}+\mathbf{A}_2\epsilon_{,\xi_2}\right]d\xi + \int_{\Gamma}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})(\epsilon^- - \epsilon)\,ds = 0. \tag{4.51}$$

The MacLaurin series of ϵ with respect to h is

$$\epsilon(\tau,\xi) = \sum_{k=0}^{p+1} h^k\mathbf{q}_k(\tau,\xi) + \mathscr{O}(h^{p+2}). \tag{4.52}$$

Since $\mathbf{e}^{\perp} - \mathbf{E}^{\perp} = \mathscr{O}(h^{p+2})$, $\mathbf{q}_k \in \mathscr{E}^p$, for $k < p+2$.

Substituting $\epsilon^-(\tau,\xi) = \mathscr{O}(h^{p+2})$ from the definition of $\mathbf{E}^-$ (4.21b) and (4.52) into (4.50) yields

$$\sum_{k=0}^{p+1} h^k\left(\iint_{\Delta}\mathbf{v}^t\left[\frac{h}{T}\mathbf{q}_{k,\tau}+\mathbf{A}_1\mathbf{q}_{k,\xi_1}+\mathbf{A}_2\mathbf{q}_{k,\xi_2}\right]d\xi - \int_{\Gamma}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_k\,ds\right) = \mathscr{O}(h^{p+2}), \tag{4.53}$$

which infers that all terms of the same power in h are zero.

For instance, the $\mathscr{O}(1)$ term leads to the orthogonality condition for $\mathbf{q}_0$

$$\iint_{\Delta}\mathbf{v}^t[\mathbf{A}_1\mathbf{q}_{0,\xi_1}+\mathbf{A}_2\mathbf{q}_{0,\xi_2}]\,d\xi - \int_{\Gamma}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_0\,ds = 0 \quad \forall\,\mathbf{v}\in\mathscr{E}^p. \tag{4.54}$$

By Lemma 4.1, $\mathbf{q}_0 = 0$.

Using induction to prove the lemma and assume that $\mathbf{q}_l = 0$, $l = 0,\dots,k-1$, and apply the $\mathscr{O}(h^k)$ term to obtain the orthogonality condition

$$\iint_{\Delta}\mathbf{v}^t[\mathbf{A}_1\mathbf{q}_{k,\xi_1}+\mathbf{A}_2\mathbf{q}_{k,\xi_2}]\,d\xi - \int_{\Gamma}\mathbf{v}^t(\nu_1\mathbf{A}_1^{\bar{\mu}_1}+\nu_2\mathbf{A}_2^{\bar{\mu}_2})\mathbf{q}_k\,ds = 0 \quad \forall\,\mathbf{v}\in\mathscr{E}^p. \tag{4.55}$$

Again, by Lemma 4.1, $\mathbf{q}_k = 0$.

Hence, $\mathbf{q}_k = 0$, $k = 0,\cdots,p+1$, i.e.,

$$\epsilon(\tau,\xi) = \mathscr{O}(h^{p+2}), \tag{4.56}$$

which completes the proof. □

We conclude this section by noting that the transient estimate $\mathbf{E}^{\maltese}(t,\mathbf{x}) \in \mathscr{N}(\mathbf{A}_1)\oplus\mathscr{N}(\mathbf{A}_2)$, hence, the stationary error estimate $\mathbf{E}^{\perp}(t,\mathbf{x})$ is accurate only for the error component lying in $(\mathscr{N}(\mathbf{A}_1)\oplus\mathscr{N}(\mathbf{A}_2))^{\perp}$.

4.3. Extension to three dimensions

Our error analysis can be extended to three-dimensional systems following the same line of reasoning. For instance, Lemmas 3.2 and 3.3 can be extended to three dimensions by considering $\mathbf{A}_i$, $i = 1,2,3$ such that either at least one matrix is invertible or if $\mathbf{A}_i$ has rank $r < m$ then one of the matrices $\mathbf{P}^t_{i,2}\mathbf{A}_j\mathbf{P}_{i,2}$, $j \neq i$ is invertible.

For smooth solutions the leading DG error term for three-dimensional problems (2.1) with $d = 3$ can be written as

$$\mathbf{e} = \mathbf{e}^{\perp} + \mathbf{e}^{\maltese} + \mathscr{O}(h^{p+2}), \tag{4.57a}$$

where

$$\mathbf{e}^{\perp}(t,h\xi) = h^{p+1}\sum_{j=1}^{3}[L_{p+1}(\xi_j)\mathbf{I} - L_p(\xi_j)sgn(\mathbf{A}_j)]\mathbf{c}_j^{\perp}(t), \tag{4.57b}$$

$$\mathbf{e}^{\maltese}(t,h\xi) = h^{p+1}\sum_{j=1}^{3}(L_{p+1}(\xi_j)\mathbf{c}_j^{\maltese}(t) - L_p(\xi_j)\mathbf{d}_j^{\maltese}(t)). \tag{4.57c}$$

The superconvergence result for every unit vector $\mathbf{z}\in\mathscr{R}(A_1^{s_1})\cap\mathscr{R}(A_2^{s_2})\cap\mathscr{R}(A_3^{s_3})$, $s_i = +,-$, can be written as

$$\mathbf{z}^t\mathbf{e}(t,h\bar{\xi}_i^{s_1},h\bar{\xi}_j^{s_1},h\bar{\xi}_k^{s_3}) = \mathscr{O}(h^{p+2}), \quad t = \mathscr{O}(1). \tag{4.58}$$

The three-dimensional stationary error estimate $\mathbf{E}^{\perp}$ is computed from (4.12a) and (4.16) for $i = 1,2,3$, while $\mathbf{E}^{\maltese}$ is computed from (4.22a) and (4.26a) for $i = 1,2,3$.

Again, we note that $\mathbf{E}^{\maltese}(t,\mathbf{x}) \in \bigoplus_{i=1}^{3}\mathscr{N}(\mathbf{A}_i)$. Thus, $\mathbf{E}^{\perp}$ should be an accurate estimate of the error component lying in $(\bigoplus_{i=1}^{3}\mathscr{N}(\mathbf{A}_i))^{\perp}$.

5. Computational examples

We validate our theory on one-, two-, and three-dimensional linear symmetric hyperbolic systems. The accuracy of *a posteriori* error estimates is measured by the local effectivity indices

$$\theta_e = \frac{\|\mathbf{E}\|_{L^2(\omega_e)}}{\|\mathbf{e}\|_{L^2(\omega_e)}} \tag{5.1}$$

and the global effectivity index with respect to the L^2 norm

$$\theta = \frac{\|\mathbf{E}\|}{\|\mathbf{e}\|}. \tag{5.2}$$

The componentwise effectivity indices are defined as

$$\theta^* = \left[\frac{\|E_1\|}{\|e_1\|},\dots,\frac{\|E_m\|}{\|e_m\|}\right]^t. \tag{5.3}$$

We also need the componentwise L^2 error

$$\|\mathbf{e}\|^* = [\|e_1\|,\dots,\|e_m\|]^t, \tag{5.4}$$

where $\mathbf{E} = [E_1,\dots,E_m]^t$ and $\mathbf{e} = [e_1,\dots,e_m]^t$.

Ideally, the effectivity indices should approach unity under mesh refinement.

5.1. Examples for superconvergence

We start with a set of examples to validate our superconvergence results of Theorem 3.1, then, we apply our error estimation procedure to several problems to show that computations and theory are in full agreement. In this manuscript we studied the spatial component of the DG discretization error only and in all our numerical computations we 4(5)-Runge–Kutta–Fehlberg method and assumed the temporal component of the error negligible. We use the L^2 projection $\Pi\mathbf{u}_0$ to approximate the initial conditions and $\pi_1,\ \pi_2$ to approximate the boundary conditions.

Example 5.1.1. We consider the linearized one-dimensional Euler equations with proper time and space scalings

$$\begin{bmatrix}\mathfrak{p}\\ u\end{bmatrix}_{,t}+\mathbf{A}\begin{bmatrix}\mathfrak{p}\\ u\end{bmatrix}_{,x}=0,\quad x\in(0,1),\quad 0<t\leqslant 1,\quad \mathbf{A}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\tag{5.5a}$$

and select the initial and boundary conditions such that the exact solution is

$$\begin{bmatrix}\mathfrak{p}\\ u\end{bmatrix}=\begin{bmatrix}\sin(t)\cos(x-1)\\-\cos(t)\sin(x-1)\end{bmatrix},\tag{5.5b}$$

where u and $\mathfrak{p}$, respectively, denote the velocity and pressure. Basic linear algebra yields that $\mathbf{z}_1=\begin{bmatrix}1\\1\end{bmatrix}\in\mathscr{R}(\mathbf{A}^+),\quad \mathbf{z}_2=\begin{bmatrix}1\\-1\end{bmatrix}\in\mathscr{R}(\mathbf{A}^-)$. Thus, applying Theorem 4.1 leads to

$$\mathbf{z}_1^t\mathbf{e}(t,\xi(x))=h^{p+1}(L_{p+1}(\xi)-L_p(\xi))\mathbf{c}(t)+\mathcal{O}(h^{p+2}),\tag{5.6}$$

$$\mathbf{z}_2^t\mathbf{e}(t,\xi(x))=h^{p+1}(L_{p+1}(\xi)+L_p(\xi))\mathbf{d}(t)+\mathcal{O}(h^{p+2}).\tag{5.7}$$

Therefore, on each element in the mesh $\mathbf{z}_1^t\mathbf{e},\mathbf{z}_2^t\mathbf{e}$, respectively, is $\mathcal{O}(h^{p+2})$ superconvergent at the shifted roots of right Radau polynomial $R_{p+1}^+(x)$ and left Radau polynomial $R_{p+1}^-(x)$.

We first solve (5.5a) on a uniform mesh having $N=6$ elements for $p=0,1,2,3$ and plot the projected true errors $\mathbf{z}_i^t\mathbf{e},\ i=1,2$, at $t=1$ versus x in Fig. 2. We observe that the error plots for $p\geqslant 1$ intersect the *x-axis* very close to Radau points marked by $\times$.

In order to show the $\mathcal{O}(h^{p+2})$ superconvergence rates under mesh refinement, we solve (5.5a) on uniform meshes having $N=10,20,30,40$ elements for $p=0,1,2,3$ and show maximum errors $|\mathbf{z}_1^t\mathbf{e}|$ and $|\mathbf{z}_2\mathbf{e}|$ at Radau points and their rates of convergence in Table 1. We observe that the maximum projected errors at Radau points are $\mathcal{O}(h^{p+2})$ superconvergent while the L^2 error is only $\mathcal{O}(h^{p+1})$. (see Table 2)

Example 5.1.2. Let us consider the two-dimensional hyperbolic system

$$\mathbf{u}_{,t}+\mathbf{A}_1\mathbf{u}_{,x}+\mathbf{A}_2\mathbf{u}_{,y}=\mathbf{f}(t,x,y),\quad (x,y)\in(0,1)^2,\quad 0<t\leqslant 1,\tag{5.8a}$$

where

$$\mathbf{A}_1=\begin{bmatrix}2&0&1\\0&0&0\\1&0&2\end{bmatrix},\quad \mathbf{A}_2=\begin{bmatrix}-5&1&0\\1&-5&0\\0&0&2\end{bmatrix}\tag{5.8b}$$

and select $\mathbf{f}(t,x,y)$, initial and boundary conditions such that the true solution is

$$\mathbf{u}=\begin{bmatrix}\exp(t-x-y)\\\exp(-t+x-y)\\\exp(-t-x+y)\end{bmatrix}.\tag{5.8c}$$

Basic linear algebra yields that $[0\ \ 1\ \ 0]^t,[1\ \ 0\ \ -1]^t,[1\ \ 0\ \ -1]^t$ are eigenvectors of $\mathbf{A}_1$ corresponding to eigenvalues 0, 1, and 3, respectively. On the other-hand, the vectors $[1,1,0]^t,[1\ \ -1\ \ 0]^t$, and $[0\ \ 0\ \ 1]^t$ are eigenvectors of $\mathbf{A}_2$ corresponding to eigenvalues $-6,-4$, and 2, respectively.

We further note that $\mathscr{R}(\mathbf{A}_1^+)=\text{span}\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_3\},\mathscr{R}(\mathbf{A}_1^-)=\{0\},\mathscr{R}(\mathbf{A}_2^+)=\text{span}\{\boldsymbol{\varepsilon}_3\}$ and $\mathscr{R}(\mathbf{A}_2^-)=\text{span}\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\}$, where $\{\boldsymbol{\varepsilon}_i,i=1,2,3\}$ is the canonical basis of $\mathbb{R}^3$ which leads to

$$\mathscr{R}(\mathbf{A}_1^+)\cap\mathscr{R}(\mathbf{A}_2^+)=\text{span}\{\boldsymbol{\varepsilon}_3\},\tag{5.9}$$

$$\mathscr{R}(\mathbf{A}_1^+)\cap\mathscr{R}(\mathbf{A}_2^-)=\text{span}\{\boldsymbol{\varepsilon}_1\}.\tag{5.10}$$

Applying Theorem 4.1 we expect $\mathcal{O}(h^{p+2})$ pointwise superconvergence of the error projections $e_1=\boldsymbol{\varepsilon}_1^t\mathbf{e}$, and $e_3=\boldsymbol{\varepsilon}_3^t\mathbf{e}$, i.e., the first

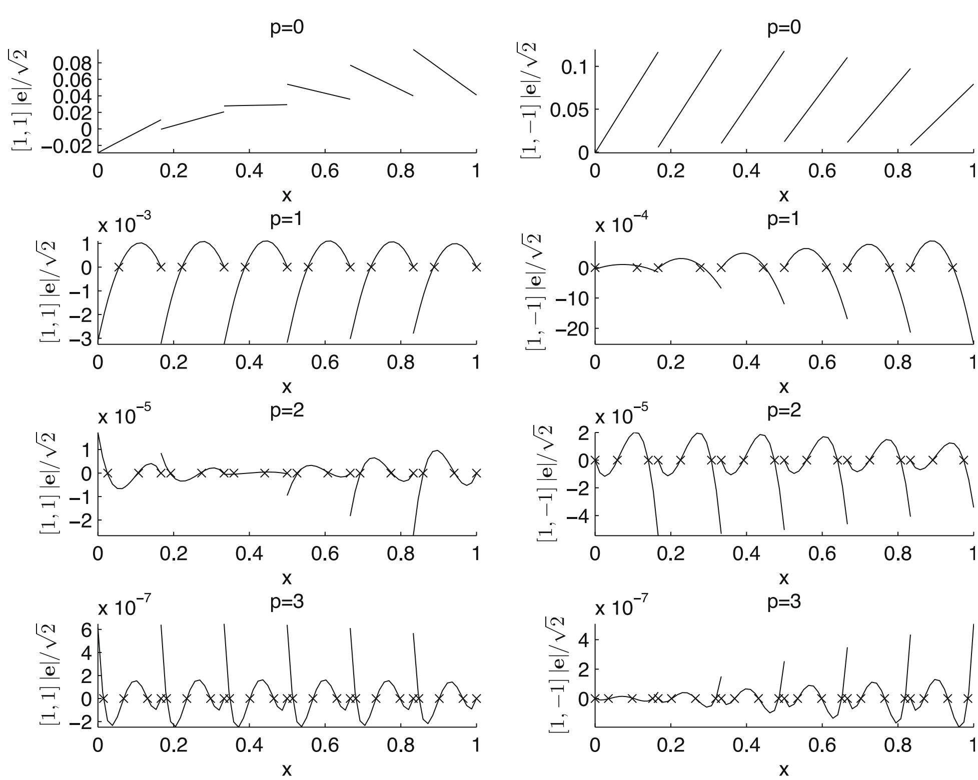

Fig. 2. Projected errors $\frac{1}{\sqrt{2}}[1,1]\mathbf{e},\frac{1}{\sqrt{2}}[1,-1]\mathbf{e}$ versus x for Example 5.1.1 at $t=1$. Shifted right Radau (left) and left Radau (right) points are marked by $\times$.

Table 1
Maximum projected errors $|[1,1]\cdot\mathbf{e}|$ at left Radau points and $|[1,-1]\cdot\mathbf{e}|$ at right Radau points and their order of convergence for Example 5.1.1 for $t=1$.

N	$p=0$		$p=1$		$p=2$		$p=3$	
	Ep	Order	Ep	Order	Ep	Order	Ep	Order
Maximum error $Ep=\|[1,1]\mathbf{e}\|$ at right Radau points								
10	$3.560e-2$	–	$2.149e-5$	–	$1.124e-7$	–	$2.546e-10$	–
20	$1.882e-2$	0.9192	$2.854e-6$	2.9128	$7.043e-9$	3.9961	$8.175e-12$	4.9610
30	$1.286e-2$	0.9404	$8.621e-7$	2.9524	$1.392e-9$	3.9979	$1.084e-12$	4.9841
40	$9.773e-3$	0.9529	$3.694e-7$	2.9455	$4.407e-10$	3.9987	$2.565e-13$	5.0092
Maximum error $Ep=\|[1,-1]\mathbf{e}\|$ *at left Radau points*								
10	$3.560e-2$	–	$2.149e-5$	–	$1.124e-7$	–	$2.546e-10$	–
20	$1.882e-2$	0.9192	$2.854e-6$	2.9128	$7.043e-9$	3.9961	$8.175e-12$	4.9610
30	$1.286e-2$	0.9404	$8.621e-7$	2.9524	$1.392e-9$	3.9979	$1.084e-12$	4.9841
40	$9.773e-3$	0.9529	$3.694e-7$	2.9455	$4.407e-10$	3.9987	$2.565e-13$	5.0092

Table 2
Maximum errors for $|e_1|$ at shifted Radau points (ξ_i^+,ξ_j^+) and $|e_3|$ at shifted Radau points (ξ_i^+,ξ_j^-) and $t=1$ for Example 5.1.2.

N	$p=0$		$p=1$		$p=2$		$p=3$	
	E	Order	E	Order	E	Order	E	Order
Maximum error $\|e_1\|$ *at Radau points*								
5^2	0.1260	–	$1.054e-3$	–	$1.704e-5$	–	$5.682e-7$	–
10^2	$6.899e-2$	0.8688	$1.622e-4$	2.6997	$1.149e-6$	3.8901	$2.156e-8$	4.7198
15^2	$4.790e-2$	0.8999	$5.128e-5$	2.8407	$2.321e-7$	3.9456	$3.036e-9$	4.8355
20^2	$3.678e-2$	0.9186	$2.272e-5$	2.8303	$7.415e-8$	3.9661	$7.450e-10$	4.8833
25^2	$2.973e-2$	0.9525	$1.199e-5$	2.8638	$3.055e-8$	3.9731	$2.491e-10$	4.9084
Maximum error $\|e_3\|$ *at Radau points*								
5^2	0.1260	–	$1.054e-3$	–	$1.704e-5$	–	$5.682e-7$	–
10^2	$6.899e-2$	0.8688	$1.622e-4$	2.6997	$1.149e-6$	3.8901	$2.156e-8$	4.7198
15^2	$4.790e-2$	0.8999	$5.128e-5$	2.8407	$2.321e-7$	3.9456	$3.036e-9$	4.8355
20^2	$3.678e-2$	0.9186	$2.272e-5$	2.8303	$7.415e-8$	3.9661	$7.450e-10$	4.8833
25^2	$2.973e-2$	0.9525	$1.199e-5$	2.8638	$3.055e-8$	3.9731	$2.491e-10$	4.9084

and third components of $\mathbf{e}$ are $\mathcal{O}(h^{p+2})$ superconvergent at Radau points.

We solve (5.8a) on a 4×4 uniform mesh for $p=1,2,3$ and plot the zero-level curves of the first and third components of the error in Fig. 3. We observe that the zero-level curves pass near shifted Radau points marked by $\times$.

In order to show superconvergence rates we solve (5.8a) on uniform square meshes having $N=5^2,10^2,15^2,20^2,25^2$ elements with $p=1,2,3$ and present the maximum errors of $|e_1|$ and $|e_3|$ at shifted Radau points over all elements and their order of convergence under mesh refinement. We observe $\mathcal{O}(h^{p+2})$ superconvergence rates which is in full agreement with Theorem 4.1.

Example 5.1.3. Let us consider the three-dimensional system

$$\mathbf{u}_{,t}+\mathbf{A}_1\mathbf{u}_{,x}+\mathbf{A}_2\mathbf{u}_{,y}+\mathbf{A}_3\mathbf{u}_{,z}=\mathbf{f}(t,x,y,z),\quad (x,y,z)^t\in(0,1)^3,\quad 0<t\leqslant 1, \tag{5.11a}$$

where

$$\mathbf{A}_1=\begin{bmatrix}2&-1&-1\\-1&1&1\\-1&1&2\end{bmatrix},\quad \mathbf{A}_2=\begin{bmatrix}-1&1&2\\1&2&1\\2&1&3\end{bmatrix},\quad \mathbf{A}_3=\begin{bmatrix}-1&0&2\\0&-1&0\\2&0&-1\end{bmatrix} \tag{5.11b}$$

and select $\mathbf{f}(t,x,y,z)$, the initial and boundary conditions such that the true solution is

$$\mathbf{u}=\begin{bmatrix}\exp(t+x-y-z)\\\exp(t-x+y-z)\\\exp(t-x-y+z)\end{bmatrix}. \tag{5.11c}$$

Basic linear algebra leads to the following results:

$$\mathcal{R}(\mathbf{A}_1^+)=\mathbb{R}^3,\mathcal{R}(\mathbf{A}_1^-)=\{0\}, \tag{5.12}$$

$$\mathcal{R}(\mathbf{A}_2^-)=\text{span}\{\mathbf{z}_1\},\mathcal{R}(\mathbf{A}_2^+)=\mathcal{R}(\mathbf{A}_2^-)^\perp,$$

$$\mathbf{z}_1=\begin{bmatrix}-0.9260656482554513\\0.1480943063089058\\0.3470885932439939\end{bmatrix} \tag{5.13}$$

and

$$\mathcal{R}(\mathbf{A}_3^+)=\text{span}\{\mathbf{z}_2\},\mathcal{R}(\mathbf{A}_3^-)=\mathcal{R}(\mathbf{A}_3^+)^\perp,\quad \mathbf{z}_2=[1,0,1]^t. \tag{5.14}$$

The spaces $\mathcal{R}(\mathbf{A}_2^-)$ and $\mathcal{R}(\mathbf{A}_3^+)$ are two planes whose intersection is the line defined as span$\{\mathbf{z}\}$, where

$$\mathbf{z}=\frac{\mathbf{z}_1\times\mathbf{z}_2}{\|\mathbf{z}_1\times\mathbf{z}_2\|_2}=\begin{bmatrix}0.1147781473323876\\0.9867380370644933\\-0.1147781473323876\end{bmatrix}\in\mathcal{R}(A_1^+)\cap\mathcal{R}(A_2^+)\cap\mathcal{R}(A_3^-). \tag{5.15}$$

Applying the superconvergence result (4.58) we show that $\mathbf{z}^t\mathbf{e}$ is $\mathcal{O}(h^{p+2})$ at the shifted Radau points $(\xi_i^+,\xi_j^+,\xi_k^-)$. Next, we solve (5.11a) on uniform meshes having $4^3,6^3,8^3,10^3$ square elements and $p=0,1,2,3$ and present in Table 3 the maximum errors $|\mathbf{z}^t\mathbf{e}|$ at shifted Radau points over all elements. These results validate the superconvergence theory of Theorem 4.1.

5.2. Examples for a posteriori error estimation

Here, we solve several examples to validate our *a posteriori* error estimation procedure where we use $\pi_i,\ i=1,2$ to approximate

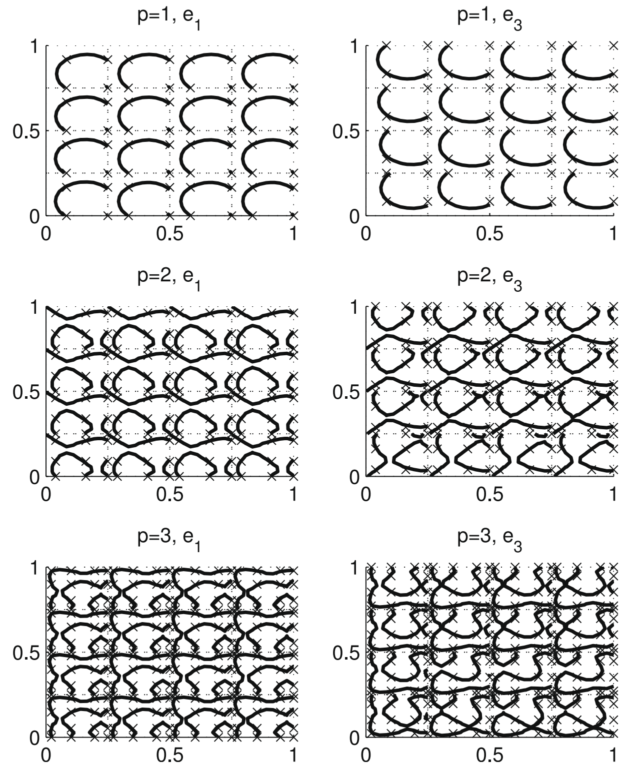

Fig. 3. Zero-level curves of e_1, e_3 at $t = 1$ for Example 5.1.2. Shifted Radau points are marked by $\times$.

the boundary conditions and both L^2 projection $\Pi\mathbf{u}_0$ and $\pi\mathbf{u}_0$ to approximate the initial conditions.

Example 5.2.1. We solve the one-dimensional Euler system (5.5a) on uniform meshes have $N = 50, 75, 100$ for $0 < t \leqslant 1$ with $p = 0, 1, 2, 3$ and compute a posteriori error estimates $\mathbf{E}^{\perp}$ by solving the stationary problem (4.12a) and (4.13). We present the L^2 errors and effectivity indices at $t = 1$ in Table 4 using Πu_0. These results indicate that the stationary error estimates $\mathbf{E}^{\perp}$ are accurate estimates of $\mathbf{e}$ for $p > 1$ which is in full agreement with the theory.

In Fig. 4 we plot the effectivity indices versus time for $N = 100, 200, 300$ and $p = 1, 2, 3$ using $\Pi\mathbf{u}_0$ (solid) and $\pi\mathbf{u}_0$ (dotted). We observe that the effectivity indices for $\Pi\mathbf{u}_0$ slightly oscillate about unity and then get closer to unity with increasing time $t > \mathcal{O}(h)$. On the other-hand, the global effectivity indices for $\pi\mathbf{u}_0$ stay close to unity at all times. Thus, we recommend the use of $\pi\mathbf{u}_0$.

Example 5.2.2. Let us consider the two-dimensional wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \quad (x, y) \in (0, 1)^2, \quad 0 < t \leqslant 1, \tag{5.16}$$

which can be written as the first-order linear hyperbolic system

$$\mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,x} + \mathbf{A}_2\mathbf{u}_{,y} = 0, \quad (x, y) \in (0, 1)^2, \quad 0 < t \leqslant 1, \tag{5.17a}$$

where

$$\mathbf{u} = \begin{bmatrix} v_{,t} + v_{,x} \\ v_{,y} \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{5.17b}$$

and select initial and boundary conditions such that the true solution is

$$\mathbf{u} = \begin{bmatrix} \sin\left(\sqrt{2}t + x + y\right) - \cos\left(-\sqrt{2}t + x + y\right) \\ \left(\sqrt{2} - 1\right)\sin\left(\sqrt{2}t + x + y\right) + \left(1 + \sqrt{2}\right)\cos\left(-\sqrt{2}t + x + y\right) \end{bmatrix}. \tag{5.17c}$$

We solve (5.17a) on uniform meshes having $N = 10^2, 20^2, 30^2$ elements for $p = 0, 1, 2, 3$ using $\Pi\mathbf{u}_0$ and present the L^2 errors and effectivity indices corresponding to the stationary error estimates $\mathbf{E}^{\perp}$ at $t = 1$ in Table 5. We observe that the effectivity indices converge to unity under mesh refinement. Furthermore, we plot the effectivity indices versus time in Fig. 5 to note that the stationary

Table 3
Maximum errors $|\mathbf{z}^t\mathbf{e}|$, $\mathbf{z}$ given by (5.15), at shifted Radau points $(\xi_i^+, \xi_j^+, \xi_k^-)$ and $t = 1$ over all elements for Example 5.1.3.

N	$p = 0$		$p = 1$		$p = 2$		$p = 3$	
	$\|\mathbf{z}^t\mathbf{e}\|$	Order	$\|\mathbf{z}^t\mathbf{e}\|$	Order	$\|\mathbf{z}^t\mathbf{e}\|$	Order	$\|\mathbf{z}^t\mathbf{e}\|$	Order
4^3	0.1320	–	$1.566e-2$	–	$6.086e-4$	–	$5.147e-5$	–
6^3	0.1067	0.5253	$5.259e-3$	2.6904	$1.403e-4$	3.6189	$7.646e-6$	4.7030
8^3	$8.419e-2$	0.8225	$2.287e-3$	2.8948	$4.808e-5$	3.7223	$1.929e-6$	4.7878
10^3	$7.472e-2$	0.5350	$1.256e-3$	2.6839	$2.066e-5$	3.7863	$6.568e-7$	4.8271

Table 4
L^2 errors $\|\mathbf{e}\|, \|\mathbf{e} - \mathbf{E}^{\perp}\|$, their rates of convergence with maximum and minimum local effectivity indices and global effectivity indices for $\mathbf{E}^{\perp}$ at $t = 1$ for Euler equations (5.5a).

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e} - \mathbf{E}^{\perp}\|$	Order	$[\min_e\theta_e, \max_e\theta_e]$	θ
0	50	$9.175e-03$	–	$3.713e-03$	–	[0.343, 1.878]	0.7945
	75	$6.187e-03$	0.972	$2.518e-03$	0.958	[0.335, 1.900]	0.7877
	100	$4.671e-03$	0.977	$1.907e-03$	0.967	[0.331, 1.914]	0.7836
1	50	$1.875e-05$	–	$2.161e-07$	–	[0.992, 1.001]	0.9997
	75	$8.338e-06$	1.999	$7.084e-08$	2.750	[0.993, 1.001]	0.9998
	100	$4.691e-06$	1.999	$3.229e-08$	2.731	[0.994, 1.001]	0.9999
2	50	$2.488e-08$	–	$1.160e-10$	–	[0.997, 1.001]	0.9999
	75	$7.369e-09$	3.001	$2.401e-11$	3.885	[0.998, 1.001]	1.0000
	100	$3.108e-09$	3.001	$7.882e-12$	3.872	[0.998, 1.001]	1.0000
3	50	$3.699e-11$	–	$7.066e-14$	–	[0.999, 1.001]	1.0000
	75	$7.309e-12$	3.999	$9.712e-15$	4.894	[0.999, 1.000]	0.9999
	100	$2.313e-12$	3.999	$3.925e-15$	3.149	[0.999, 1.000]	0.9998

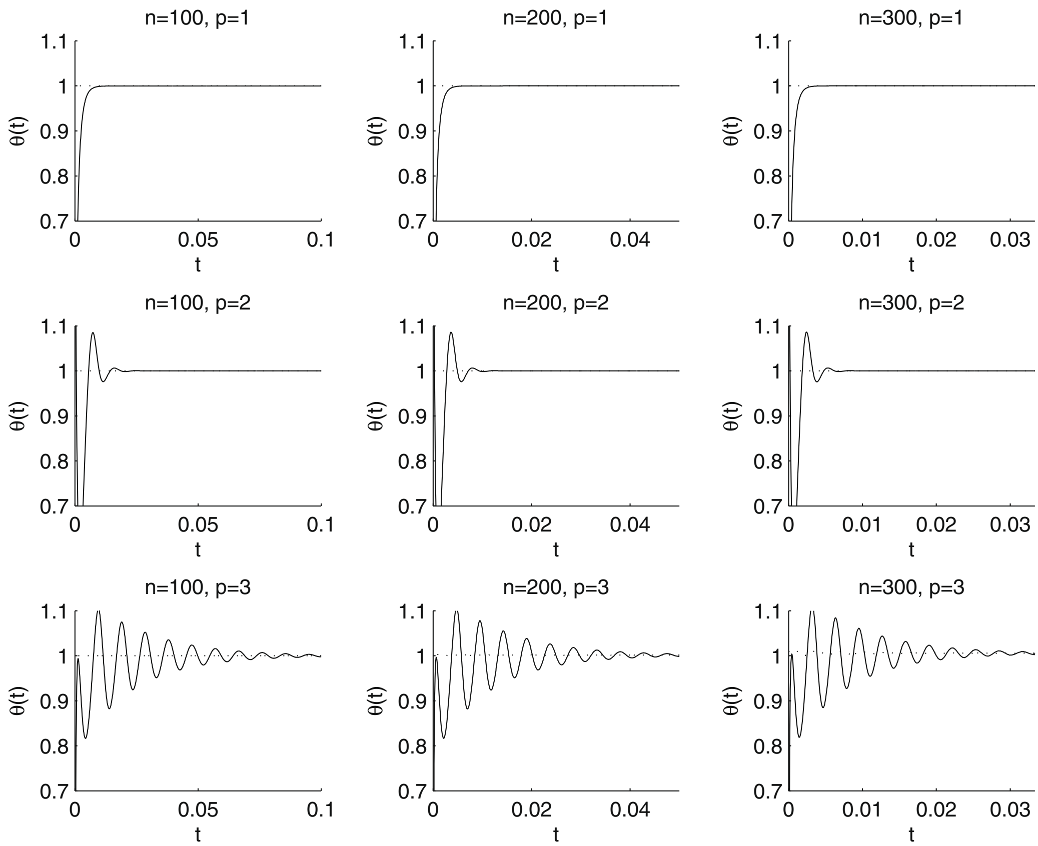

Fig. 4. Global effectivity indices versus t over the interval $[0, \frac{10}{h}]$ for $N = 100, 200, 300, p = 1, 2, 3$ using $\pi\mathbf{u}_0$ (dotted) and $\Pi\mathbf{u}_0$ (solid) for Euler equations (5.5a).

Table 5
L^2 errors $\|\mathbf{e}\|, \|\mathbf{e} - \mathbf{E}^{\perp}\|$ and their order of convergence. Maximum, minimum local and global effectivity indices for problem (5.17a) at $t = 1$ using $\Pi\mathbf{u}_0$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e} - \mathbf{E}^{\perp}\|$	Order	$[\min_e\theta_e, \max_e\theta_e]$	θ
0	10^2	$1.380e-01$	–	$1.073e-01$	–	[0.180, 2.872]	0.780
	20^2	$7.322e-02$	0.914	$5.760e-02$	0.898	[0.165, 3.375]	0.771
	30^2	$4.997e-02$	0.942	$3.957e-02$	0.925	[0.157, 3.490]	0.766
1	10^2	$2.062e-03$	–	$1.039e-04$	–	[0.966, 1.007]	0.994
	20^2	$5.119e-04$	2.010	$1.478e-05$	2.813	[0.980, 1.005]	0.998
	30^2	$2.270e-04$	2.006	$4.772e-06$	2.789	[0.986, 1.004]	0.999
2	10^2	$1.059e-05$	–	$9.583e-07$	–	[0.966, 1.015]	1.002
	20^2	$1.331e-06$	2.992	$6.093e-08$	3.975	[0.985, 1.009]	1.002
	30^2	$3.953e-07$	2.995	$1.211e-08$	3.984	[0.991, 1.006]	1.001
3	10^2	$1.008e-07$	–	$4.781e-09$	–	[0.978, 1.006]	0.999
	20^2	$6.282e-09$	4.004	$1.512e-10$	4.982	[0.992, 1.003]	1.000
	30^2	$1.240e-09$	4.002	$2.000e-11$	4.989	[0.996, 1.002]	1.000

error estimate $\mathbf{E}^{\perp}$ is asymptotically accurate which is in full agreement with Theorem 4.2. We further note that the effectivity indices stay close to unity at all times when using $\pi\mathbf{u}_0$. For $\Pi\mathbf{u}_0$ the effectivity indices oscillates about unity near $t = 0$ before approaching unity.

Example 5.2.3. Let us consider three-dimensional wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \quad (x, y, z) \in (0.1)^3, \quad 0 < t \leqslant 1, \tag{5.18}$$

which can be written as the first-order linear hyperbolic system

$$\mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,x} + \mathbf{A}_2\mathbf{u}_{,y} + \mathbf{A}_3\mathbf{u}_{,z} = 0, \quad (x, y, z) \in (0, 1)^3, \quad 0 < t \leqslant 1, \tag{5.19a}$$

where

$$\mathbf{u} = \begin{bmatrix} v_{,t} + v_{,x} \\ v_{,y} \\ v_{,z} \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{5.19b}$$

and select initial and boundary conditions such that the true solution is

$$\mathbf{u} = \left[2, \left(\sqrt{3} - 1\right), \left(\sqrt{3} - 1\right)\right]^t \sin\left(\sqrt{3}t + x + y + z\right). \tag{5.19c}$$

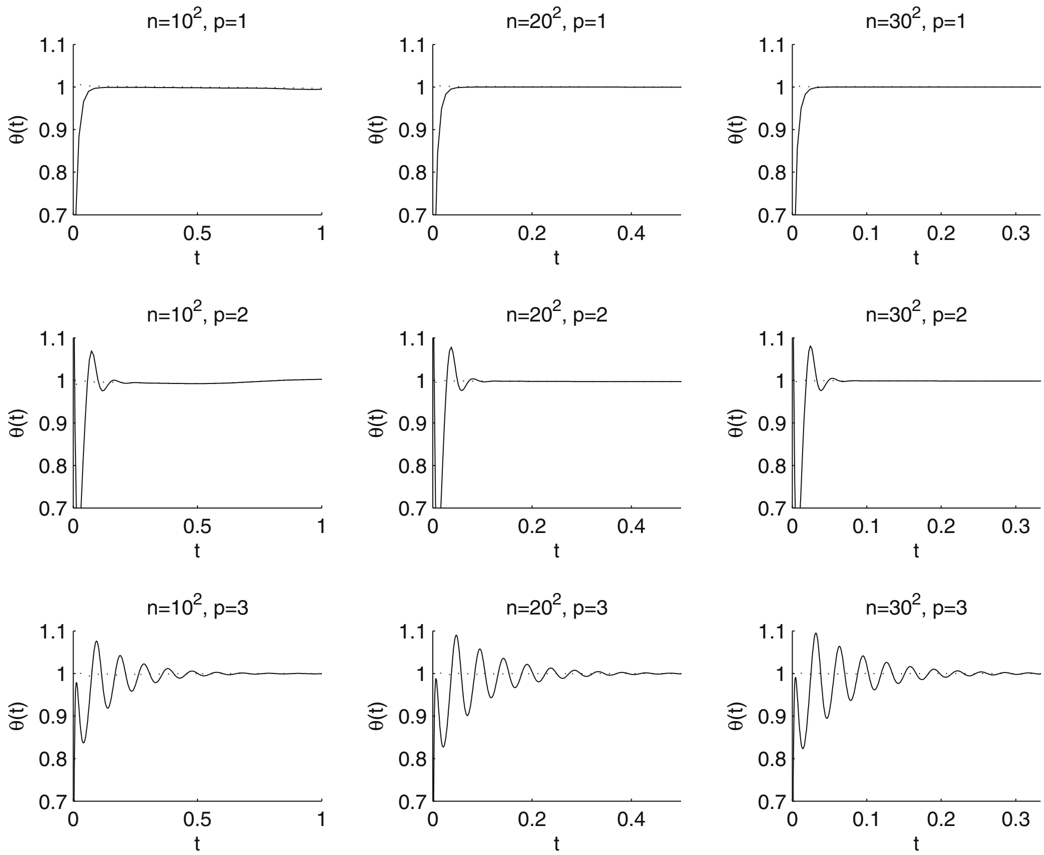

Fig. 5. Global effectivity indices versus $0 \leqslant t \leqslant \frac{10}{h}$ using $\pi\mathbf{u}_0$ (dotted) and $\Pi\mathbf{u}_0$ (solid) for problem (5.17a).

Table 6
L^2 errors $\|\mathbf{e}\|, \|\mathbf{e} - \mathbf{E}^{\perp} - \mathbf{E}^{\maltese}\|$ and their order of convergence. Global effectivity indices corresponding to transient estimates for problem (5.19a) at $t = 1$ using $\Pi\mathbf{u}_0$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e} - \mathbf{E}^{\perp} - \mathbf{E}^{\maltese}\|$	Order	θ
1	10^3	$1.0690e-3$	–	$1.7744e-4$	–	0.9897
	15^3	$4.7450e-4$	2.003	$5.4945e-5$	2.891	0.9949
	20^3	$2.6690e-4$	2	$2.4042e-5$	2.873	0.9969
	30^3	$1.1867e-4$	1.999	$7.5900e-6$	2.844	0.9985
2	10^3	$1.6675e-5$	–	$1.0085e-6$	–	0.998
	15^3	$4.9355e-6$	3.003	$2.0831e-7$	3.89	0.9988
	20^3	$2.0813e-6$	3.001	$6.8713e-8$	3.855	0.9992
	30^3	$6.1650e-7$	3.001	$1.4723e-8$	3.799	0.9994
3	10^3	$5.9039e-8$	–	$2.7592e-8$	–	0.8923
	15^3	$1.0998e-8$	4.145	$3.6731e-9$	4.973	0.9463
	20^3	$3.4026e-9$	4.078	$8.7744e-10$	4.977	0.9684

While the matrix $\mathbf{A}_1$ is invertible and admits the eigenvalues $\{-1, 1, 1\}$, both $\mathbf{A}_2, \mathbf{A}_3$ are singular and admit the eigenvalues $\{-1, 0, 1\}$. Moreover, the eigenvectors vectors $[0, 0, 1]^t$ and $[0, 1, 0]^t$ are associated with the zero eigenvalue for $\mathbf{A}_2$ and $\mathbf{A}_3$, respectively. Applying our theory, the stationary error estimate $\mathbf{E}^{\perp}$ can only accurately estimate the component of error lying in $(\mathcal{N}(\mathbf{A}_1) \oplus \mathcal{N}(A_2) \oplus \mathcal{N}(A_3))^{\perp} = \text{span}\{[1, 0, 0]^t\}$, i.e., only $E_1^{\perp}$ is an accurate estimate of e_1.

In order to validate our theory we solve (5.19a) on uniform meshes having $N = 10^3, 15^3, 20^3, 30^3$ elements with $p = 1, 2, 3$ and show the stationary error estimates at $t = 1$ in Table 7. These results confirm our theory, i.e., only $E_1^{\perp}$ is an accurate estimate of e_1. On the other-hand we observe that the transient error estimates $\mathbf{E}^{\perp} + \mathbf{E}^{\maltese}$ shown in Table 6 are accurate for all components of $\mathbf{e}$ and converge to the true error with mesh refinement.

Finally, we plot the global effectivity indices versus time in Fig. 6 where we observe that the global effectivity indices for the transient *a posteriori* error estimate $\mathbf{E}^{\perp} + \mathbf{E}^{\maltese}$ oscillates near $t = 0$ before approaching unity with increasing time for $\Pi\mathbf{u}_0$. However, effectivity indices stay close to unity at all times when using $\pi\mathbf{u}_0$.

6. Conclusion

We investigated the superconvergence properties of the DG method applied to symmetric linear hyperbolic systems. We established pointwise and averaged superconvergence results. We explicitly write the leading term of the DG finite element error on each element and constructed asymptotically exact *a posteriori* error estimate under mesh refinement. In the near future we plan

Table 7
Componentwise L^2 errors $\|\mathbf{e}\|^*$, $\|\mathbf{e}-\mathbf{E}^{\perp}\|^*$ at $t=1$, their order of convergence and global effectivity indices θ^* corresponding to stationary estimate for problem (5.19a) using $\Pi\mathbf{u}_0$.

p	N	$\|\mathbf{e}\|^*$	Order	$\|\mathbf{e}-\mathbf{E}^{\perp}\|^*$	Order	θ^*
1	10^3	$\begin{bmatrix}7.7279e-4\\5.2229e-4\\5.2229e-4\end{bmatrix}$	–	$\begin{bmatrix}1.4936e-4\\1.7895e-4\\1.7895e-4\end{bmatrix}$	–	$\begin{bmatrix}0.9854\\0.9414\\0.9414\end{bmatrix}$
	15^3	$\begin{bmatrix}3.4175e-4\\2.3277e-4\\2.3277e-4\end{bmatrix}$	$\begin{bmatrix}2.0124\\1.9932\\1.9932\end{bmatrix}$	$\begin{bmatrix}4.5616e-5\\7.7992e-5\\7.7992e-5\end{bmatrix}$	$\begin{bmatrix}2.9253\\2.0483\\2.0483\end{bmatrix}$	$\begin{bmatrix}0.9930\\0.9430\\0.9430\end{bmatrix}$
	20^3	$\begin{bmatrix}1.9191e-4\\1.3116e-4\\1.3116e-4\end{bmatrix}$	$\begin{bmatrix}2.0058\\1.9939\\1.9939\end{bmatrix}$	$\begin{bmatrix}1.9691e-5\\4.3639e-5\\4.3639e-5\end{bmatrix}$	$\begin{bmatrix}2.9202\\2.0184\\2.0184\end{bmatrix}$	$\begin{bmatrix}0.9958\\0.9435\\0.9435\end{bmatrix}$
	30^3	$\begin{bmatrix}8.5201e-5\\5.8411e-5\\5.8411e-5\end{bmatrix}$	$\begin{bmatrix}2.0027\\1.9951\\1.9951\end{bmatrix}$	$\begin{bmatrix}6.0422e-6\\1.9367e-5\\1.9367e-5\end{bmatrix}$	$\begin{bmatrix}2.9137\\2.0036\\2.0036\end{bmatrix}$	$\begin{bmatrix}0.9980\\0.9436\\0.9436\end{bmatrix}$
2	10^3	$\begin{bmatrix}1.2286e-5\\7.9715e-6\\7.9715e-6\end{bmatrix}$	–	$\begin{bmatrix}7.6622e-7\\2.2178e-6\\2.2178e-6\end{bmatrix}$	–	$\begin{bmatrix}0.9987\\0.9599\\0.9599\end{bmatrix}$
	15^3	$\begin{bmatrix}3.6364e-6\\2.3596e-6\\2.3596e-6\end{bmatrix}$	$\begin{bmatrix}3.0027\\3.0024\\3.0024\end{bmatrix}$	$\begin{bmatrix}1.5291e-7\\6.5430e-7\\6.5430e-7\end{bmatrix}$	$\begin{bmatrix}3.9747\\3.0106\\3.0106\end{bmatrix}$	$\begin{bmatrix}0.9993\\0.9604\\0.9604\end{bmatrix}$
	20^3	$\begin{bmatrix}1.5335e-6\\9.9509e-7\\9.9509e-7\end{bmatrix}$	$\begin{bmatrix}3.0014\\3.0013\\3.0013\end{bmatrix}$	$\begin{bmatrix}4.8655e-8\\2.7577e-7\\2.7577e-7\end{bmatrix}$	$\begin{bmatrix}3.9804\\3.0034\\3.0034\end{bmatrix}$	$\begin{bmatrix}0.9995\\0.9606\\0.9606\end{bmatrix}$
	30^3	$\begin{bmatrix}4.5422e-7\\2.9475e-7\\2.9475e-7\end{bmatrix}$	$\begin{bmatrix}3.0008\\3.0008\\3.0008\end{bmatrix}$	$\begin{bmatrix}9.6651e-9\\8.1706e-8\\8.1706e-8\end{bmatrix}$	$\begin{bmatrix}3.9861\\3.0001\\3.0001\end{bmatrix}$	$\begin{bmatrix}0.9997\\0.9607\\0.9607\end{bmatrix}$
3	10^3	$\begin{bmatrix}4.3963e-8\\2.7864e-8\\2.7864e-8\end{bmatrix}$	–	$\begin{bmatrix}2.1158e-8\\1.5094e-8\\1.5094e-8\end{bmatrix}$	–	$\begin{bmatrix}0.8856\\0.8500\\0.8500\end{bmatrix}$
	15^3	$\begin{bmatrix}8.1367e-9\\5.2316e-9\\5.2316e-9\end{bmatrix}$	$\begin{bmatrix}4.1606\\4.1252\\4.1252\end{bmatrix}$	$\begin{bmatrix}2.7924e-9\\2.3889e-9\\2.3889e-9\end{bmatrix}$	$\begin{bmatrix}4.9945\\4.5465\\4.5465\end{bmatrix}$	$\begin{bmatrix}0.9441\\0.8943\\0.8943\end{bmatrix}$
	20^3	$\begin{bmatrix}2.5112e-9\\1.6234e-9\\1.6234e-9\end{bmatrix}$	$\begin{bmatrix}4.0865\\4.0676\\4.0676\end{bmatrix}$	$\begin{bmatrix}6.6341e-10\\6.7499e-10\\6.7499e-10\end{bmatrix}$	$\begin{bmatrix}4.9960\\4.3934\\4.3934\end{bmatrix}$	$\begin{bmatrix}0.9678\\0.9123\\0.9123\end{bmatrix}$

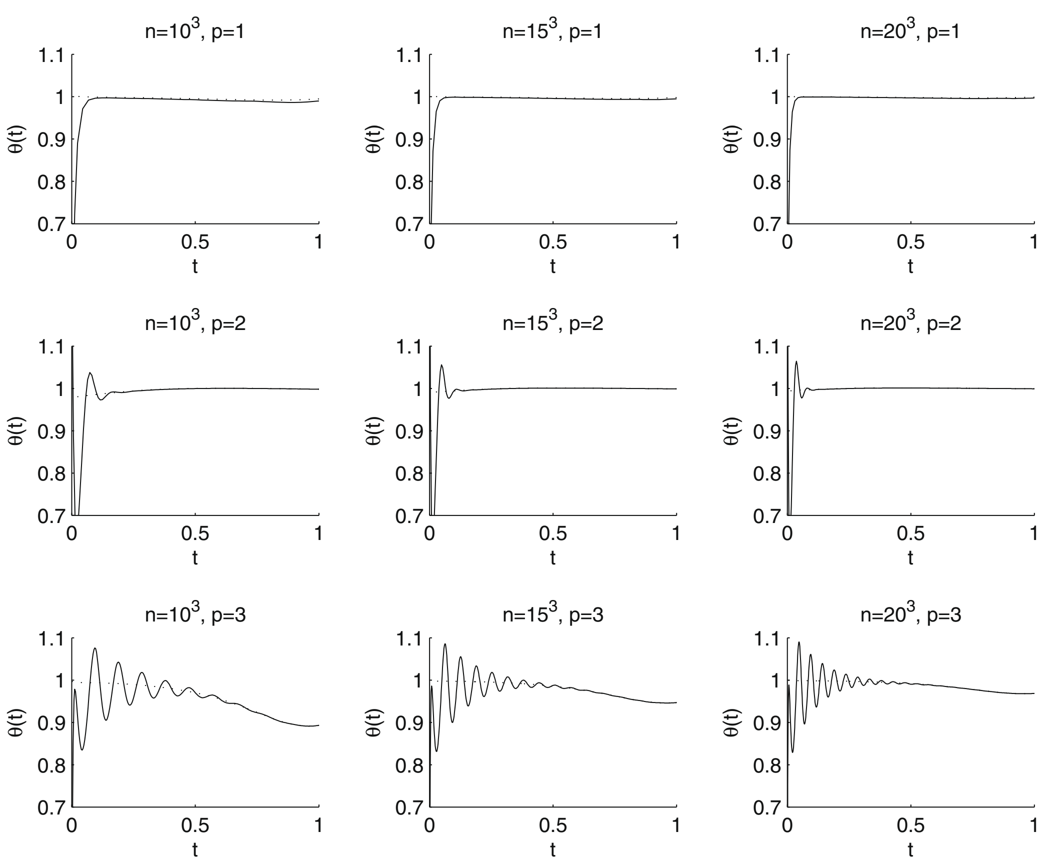

Fig. 6. Global effectivity indices versus time using $\pi\mathbf{u}_0$ (dotted) and $\Pi\mathbf{u}_0$ (solid) for problem (5.19a).

to extended these results to the more general linear and nonlinear hyperbolic systems. We also plan to investigate the extension of the work of Adjerid and Baccouch [2,3] on triangular meshes to hyperbolic systems. Currently, we are investigating the behavior of DG errors when other numerical fluxes such Lax-Friedrichs are used and we will report our findings in a forthcoming paper. Since our theory does not hold near singularities, in the future we will apply our techniques to solve linear and nonlinear problems with discontinuous solutions and test the scope of applicability of our results.

Acknowledgement

This research was partially supported by the National Science Foundation (Grant Number DMS 0511806).

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